FIFTH-GRADE STUDENTS' LEARNING OF MULTIPLICATION OF FRACTIONS BASED ON REALISTIC MATHEMATICS EDUCATION

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ABSTRACT

FIFTH-GRADE STUDENTS' LEARNING OF MULTIPLICATION OF FRACTIONS BASED ON REALISTIC MATHEMATICS EDUCATION

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This study investigated fifth-grade students' learning of multiplication of fractions designed based on Realistic Mathematics Education (RME). The study used design research with a Hypothetical Learning Trajectory (HLT) as a research instrument. The HLT was tested with five students at an international school in Ankara, Türkiye (i.e., two students at the pilot experiment and three students at the teaching experiment). The study found that using well-chosen contexts helped students use models for multiplication of fractions, leading to formalization and number sense acquisition. The findings suggested that students' learning of multiplication of fractions improved through mathematizing processes. These included modeling (when attempting to solve problems, students utilized models like number lines, ratio tables, and arrays); symbolizing (for example, in the context of $\frac{3}{4}$ of a lot representing the playground, the "4" indicates that the rectangle should be divided into four equal parts, and the "3" represents that three of these parts should be shaded to visually depict the portion of the playground belonging to the lot/garden as a whole); generalizing (as students made a generalization that the word "of" signifies multiplication, they discovered that $\frac{2}{5}$ of $\frac{3}{4}$

means $\frac{2}{5} \times \frac{3}{4}$); formalizing (when students noticed a pattern that involved multiplying a fraction by a whole number, they could add 1 to the denominator of the whole number, and then, they would simply multiply the two fractions); and abstracting (when students applied their mathematical skills to solve the problem more formally). Additionally, the study identified the challenges that students faced when learning to multiply fractions using the designed activities. The research findings can be utilized as a valuable reference for developing effective instructional activities for teaching and learning the multiplication of fractions.

Keywords: design research, Hypothetical Learning Trajectory, mathematizing, multiplication of fractions, Realistic Mathematics Education

ÖZ

BEŞİNCİ SINIF ÖĞRENCİLERİNİN GERÇEKÇİ MATEMATİK EĞİTİMİNE DAYALI KESİRLERDE ÇARPMA İŞLEMİNİ ÖĞRENMELERİ

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Bu çalışma, Gerçekçi Matematik Eğitimi (RME) temel alınarak tasarlanan kesirlerin çarpımını beşinci sınıf öğrencilerinin öğrenmesini araştırmıştır. Çalışmada, Araştırma Enstrümanı olarak bir Varsayımsal Öğrenme Yörüngesi (HLT) ile araştırma tasarımı kullanıldı. HLT, Türkiye, Ankara'daki bir uluslararası okulundan seçilmiş beş öğrenci ile test edilmiştir (yani, iki öğrenci pilot deneyde ve üç öğrenci öğretim deneyinde kullanıldı). Çalışma, iyi seçilmiş bağlamların kullanımının, öğrencilerin kesirlerin çarpımı için modelleri kullanmalarına ve bu sayede biçimlendirme ve sayı duyusu kazanmalarına yardımcı olduğunu bulmuştur. Bulgular, öğrencilerin kesirlerin çarpımını öğrenmelerinin matematikleştirme işlemleri yoluyla geliştiğini göstermiştir. Bu işlemler arasından, modelleme (öğrenciler, problemler çözmeye çalışırken sayı doğrusu, oran tablosu ve dizi modellerini kullandılar); simgeleştirme (örneğin, $\frac{3}{4}$'lük bir alanın oyun parkı olduğu bağlamında, "4" sembolü dikdörtgenin dört parçaya bölünmesi gerektiğini sembolize ederken, "3" sembolü bu parçaların üçünün çizilerek alan/bahçe, bütünden ayrılmış olan oyun parkının bölümünü gösterdiğini sembolize

ettiğini buldular); genelleştirme (öğrencilerin "ün" sözcüğünün çarpım işlemini ifade ettiği genelleştirmesini yaptıkça, $\frac{3}{4}$'ün $\frac{2}{5}$'ini $\frac{2}{5} \times \frac{3}{4}$ olduğunu buldular); biçimlendirme (öğrencilerin bir kesri bir doğal sayı ile çarpmayı içeren bir düzen fark ettiklerinde, tam doğal sayının paydasına 1 ekleyerek iki kesri basitçe çarpabildikleri buldular); ve soyutlama (öğrencilerin problemleri daha matematiksel bir şekilde çözmek için becerilerini kullandılar) gibi aşamalardan oluşmaktadır. Ayrıca, çalışmayı, öğrencilerin tasarlanan etkinlikleri kullanarak kesirleri çoğaltmayı öğrenirken karşılaştıkları zorlukları tespit etmiştir. Araştırma bulguları, kesirlerin çoğalmasını öğretmek ve öğrenmek için etkili öğretim faaliyetleri geliştirmek için değerli bir referans olarak kullanılabilir.

Anahtar Kelimeler: Gerçekçi Matematik Eğitimi, kesirlerde çarpma, matematikleştirme, tasarım araştırması, Varsayımsal Öğrenme Yörüngesi

To My Family

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LIST OF ABBREVIATIONS

CCSSM	Common Core State Standards for Mathematics
CCSSI	Common Core State Standards Initiative
CCSWT	Common Core Standards Writing Team
HLT	Hypothetical Learning Trajectory
NCTM	National Council of Teachers of Mathematics
NGA	National Governors Association
NMAP	National Mathematics Advisory Panel
PISA	Programme for International Student Assessment
RME	Realistic Mathematics Education

CHAPTER 1

INTRODUCTION

The topic of fractions has always been considered one of the most challenging and difficult subjects for students, as various studies have shown (Barash & Klein, 1996; Aksu, 1997; Mack, 2000; Van de Walle et al., 2013). If students do not have a proper understanding of fraction concepts, as noted by Brown and Quinn (2007), they may struggle with computations involving fractions, decimals, and percentages, as well as applying fractions in other areas, such as algebra.

It is common for students to find it hard to comprehend the logic behind fractional computations and they may end up memorizing formulas and procedures instead of learning core concepts (Murray & Newstead, 1998; De Castro, 2004; Işıksal & Çakiroğlu, 2011). One reason for this difficulty is that not all natural number properties apply to fractions, leading to confusion (Prediger, 2008). An example of this is the product of multiplying fractions, which may be larger or smaller than the multiplicand. Convincing students that multiplication makes things bigger is one of the primary obstacles in solving the problem of fraction multiplication comprehension (Vamvakoussi & Vosniadou, 2010).

Several researchers emphasize that comprehension of fraction language is a prerequisite for the development of precise and extensive fraction concepts (Streefland, 1991; Bezuk & Bieck, 1993; Connell & Peck, 1993; Mack, 1995; Mack, 2000). It has also been shown that students tend to get confused during formal fraction operations if their fraction language is inconsistent (Clements, 1980; Hunting, 1983). Understanding fraction-related concepts is essential as a lack of understanding can lead to arithmetic anxiety and limit future mathematical and scientific engagement (Gabriel et al., 2013).

According to Siegler et al. (2011) and Tian & Siegler (2017), students who have not yet understood fractions tend to assume that the properties of natural numbers apply to all numbers. This bias towards natural numbers, identified by Ni and Zhou (2005), makes it difficult for students to understand fractions because they use the characteristics of natural numbers to make assumptions about rational numbers. As a result, students struggle to see natural numbers as divisible units.

Vamvakoussi & Vosniadou (2010) argued that rational numbers and natural numbers are two distinct categories of numbers with significant mathematical differences. An infinite number of rational numbers exist between any two rational numbers, but no natural number exists between two natural numbers (Vamvakoussi & Vosniadou, 2010). Furthermore, an infinite number of fractions can represent rational numbers. This is similar to the concept of equivalent fractions. Fraction symbols are represented as $\frac{a}{b}$. Students often view the numerator and denominator as separate natural numbers (Pitkethly & Hunting, 1996) and apply processes only applicable to natural numbers (Nunes & Bryant, 1996).

As a result, students commonly make mistakes when adding and subtracting fractions (e.g., $\frac{1}{6} + \frac{1}{2} = \frac{1}{8}$) and when comparing fractions (e.g., $\frac{1}{6} > \frac{1}{2}$). Students may reason that if the number is larger, its magnitude is larger. However, with fractions, a larger denominator indicates a smaller value. Multiplying fractions also presents challenges. When multiplying natural numbers, the result is always greater than when multiplying fractions (e.g., $10 \times \frac{1}{5} = 2$).

Incorrect generalizations about natural numbers are more resistant because they come before the understanding of rational numbers (Vamvakoussi and Vosniadou, 2004). To avoid these errors, students may need to undergo conceptual restructuring in which they assimilate rational numbers as a new category of numbers with their own rules and functions (Stafylidou and Vosniadou, 2004). Additionally, natural number knowledge often dominates students' processing of fractions (Bonato et al., 2007; Kallai & Tzelgov, 2009).

The concept of fractions is also challenging (Kieren, 1993; Brousseau et al., 2004). Kieren (1976) was the first to categorize fractions as ratios, operators, quotients, and measurements. The ratio category conveys the idea of comparing two quantities. For example, if there are four boys for every five girls in a group, the ratio of boys to girls is 4:5. Boys make up $\frac{4}{9}$ of the group and girls make up $\frac{5}{9}$.

In the operator category, fractions are considered functions applied to objects, numbers, or sets (Behr et al., 1992). The fraction operator can be used to increase or decrease a quantity to a new value. For example, $\frac{3}{4}$ of a number can be found by multiplying by 3 and dividing by 4, or by dividing by 4 and then multiplying by 3. The quotient category represents the result of division. For example, the fraction $\frac{3}{4}$ can be viewed as a quotient. In the measurement category, fractions are related to two interrelated concepts. They are first seen as natural numbers representing the size of fractions. Second, they are associated with interval measurement. According to Kieren (1993), these four categories involve the part-whole concept of fractions.

Behr et al. (1992) proposed a theoretical model that linked the different categories of fractions. They suggested treating part-whole as a separate category and associating partitioning with the part-whole concept. Partitioning a set of discrete items would be equivalent to dividing a continuous quantity into equal parts (e.g., a pizza) (e.g., distributing the same number of candies among a group of children).

Cramer and Bezuk (1991) and Simon et al. (2018) found that students can multiply two fractions without understanding the meaning of the fractions being multiplied or the result. Kennedy and Tipps (1997) reported that students often struggle to conceptualize the process of multiplying fractions. This is evidenced by their difficulty in recognizing word problems involving fractions that can be solved using multiplication (Mack, 2000; Simon et al., 2018). Simon (1993) found that when pre-service teachers were asked to create a word problem based on the numerical sentence $\frac{1}{2} \times \frac{3}{4}$, 36% of them came up with problems that could be solved using a different mathematical operation, most often multiplication. Prediger (2008) found the same with seventh-grade students. She also noted the difficulty students had in understanding the relationship between the size of a product (greater or less than) and its factors. Students' difficulty in finding the correct reference unit for each number in a word problem is further evidence that they do not understand fraction multiplication (Mack, 2000; Izsák, 2008; Prediger, 2008; Hackenberg & Tillema, 2009; Webel & DeLeeuw, 2016).

Rule and Hallagan (2006) found that multiplication and division by fractions are among the most challenging topics in elementary mathematics, and many teachers and students do not have a strong understanding of these concepts. Students who have a conceptual understanding of models may be better equipped to learn fraction multiplication and division. Models can help teachers understand their students' individual understandings and discuss mathematical relationships and concepts (Goldin & Kaput, 1996). Models can facilitate the development, communication, and expression of mathematical ideas. If teachers have a deeper understanding of their students' ideas, they may be able to use student work as models to create more student-centered classrooms (Kalathil & Sherin, 2000). Models are important in mathematics education because they support both the learning of mathematics and the study of mathematics education.

Models can have many representational formats, and the relevance of a model often depends on its context of use. A representation is a configuration that "...relates to, stands for, denotes, interacts uniquely with, or otherwise represents another entity" (Goldin & Kaput, 1996, p. 398). According to the National Council of Teachers of Mathematics (NCTM), the term representation refers to both processes and products. The former refers to capturing a specific concept or idea, while the latter refers to the chosen form of representation for that concept or idea (Goldin, 2003).

Individual models do not arise spontaneously; the construction of representations is influenced by other thoughts and ideas. Representations can change or evolve as a person incorporates new information and experiences into their world model (Goldin

& Kaput, 1996). Sometimes the terms model and representation are used interchangeably. According to Van de Walle et al. (2008), a model is "any object, picture, or drawing that represents a concept or onto which the relationships for that concept can be imposed" (p. 27). Models can serve as communication tools. Zazkis and Liljedahl (2004) found that models facilitate the exchange of ideas and constructive interactions, promoting mathematical discourse. Models can also aid problem-solving by allowing students to focus on symbol manipulation before addressing the meaning of the result.

Students can only benefit from models if they can make the connection between the actual ideas and the intended ideas being represented (Zazkis & Liljedahl, 2004). Modeling is an essential learning step before considering computational methods. Due to the fragility of computational skills, "the use of models to demonstrate the meaning of division should precede the learning of an algorithm for fractional division" (Petit et al., 2010, p. 8). In contrast, students are much more likely to remember models based on a deep understanding in the future.

The ability of students to reason about fraction operations is a critical component in developing a deep understanding of mathematical concepts. The National Mathematics Advisory Panel (NMAP, 2008) found that having a strong understanding of fractions is essential for students to advance to algebra. Research has suggested that understanding fractions is a good predictor of mathematical proficiency in algebra (Bailey et al., 2012; Siegler et al., 2012; Gunderson, et al., 2019). However, despite its importance, understanding fraction operations is challenging for both young students (Siegler et al., 2011) and adults (Luo et al., 2011). Although students learn about fractions in first grade, expressing fractions and performing fraction operations remain problematic for many students (Siegler et al., 2011; Siegler & Pyke, 2013; Lortie-Forgues et al., 2015; Sidney & Alibali, 2015, 2017; Reinhold et al., 2020).

Visualizations and other forms of representation are often used in lessons and problem-solving activities to help students understand complex fraction-related concepts. The IES Practice Guide for Effective Fraction Instruction (Siegler et al., 2010) recommends using visual representation to engage students in meaningful activities and provide a strong foundation for understanding fraction concepts and operations. Teachers are encouraged to use visual representations to help students understand fractions.

According to the Common Core Standards Writing Team (2013), visual representations should be used to introduce students to fractions in first and second grade under the Geometry strand. In third grade, students will be introduced to the number line and taught to represent fractions on it using partitioning. In fourth and fifth grade, students will use visual representations when learning to multiply and divide fractions.

Numerous effective intervention methods have been studied and tested empirically for fractions, such as the Rational Number Project Curriculum and others like Fazioet al. (2016), Fuchs et al. (2013), Kellman et al. (2008), Moss and Case (1999), as well as Rau et al. (2014). Visual models constitute a fundamental component of many of these interventions. However, there is significant variation in how and which visual models are included, with some, like Rau et al. (2014) and Sidney, et al (2019), using multiple types of visual models, which raises concerns about whether certain types of visual models are more effective in supporting students' fractional reasoning than others.

In light of the aforementioned issues, it is necessary to develop fraction-learning instruction in order to enhance students' fraction knowledge. According to some researchers, this could involve the use of rich contexts, models/representations, and manipulatives, estimation, incorporating students' contributions from problem-solving skills, and addressing their misconceptions about fractions (Streefland, 1991; Arcavi, 1994; Murray & Newstead, 1998; Keijzer, 2003; Siegler et al., 2010).

According to Keijzer (2003), formal mathematics is frequently taught in elementary school. In recent decades, he asserted, numerous mathematics educators have worked to redefine mathematics education by viewing mathematics as a productive activity. In addition, research indicates that traditional teaching methods, which rely on repetitive exercises and procedures, are not effective in helping students gain a deep understanding and application of fractions; thus, alternative approaches are needed (Putri & Dolk, 2017).

Enter Realistic Mathematics Education (RME), an innovative instructional approach that strives to link mathematics education with real-world contexts to promote meaningful and conceptual learning. RME takes a problem-solving approach and encourages learners to participate in mathematically rich tasks through which they can explore concepts, discover patterns, and generate conjectures (Freudenthal, 1991; Streefland, 1991; Gravemeijer, 1994; Keijzer, 2003). RME aligns with contemporary mathematical learning theories, such as constructivist learning theory and socio-cultural theory, which contend that students learn through dynamic interaction with the subject material, social discourse with peers and teachers, and assimilation into their socio-cultural context (Gravemeijer, 1994; Keijzer, 2003).

This education approach originated in the 1970s in the Netherlands and entailed investigating rich situations, followed by a mathematizing process that leads to abstract mathematical concepts (Treffers, 1987; Streefland, 1991; Keijzer, 2003). It has been identified as a promising approach in teaching mathematics that focuses on contextualizing mathematical concepts in real-world problem-solving situations. The approach emphasizes the importance of mathematizing, which involves the process of translating a real-world problem into a mathematical problem and then solving it using appropriate mathematical procedures. RME has been shown to improve students' understanding of mathematical concepts and their problem-solving skills (Van den Heuvel-Panhuizen, 2010).

Previous studies have shown the potential benefits of RME in teaching fractions, such as enhancing students' conceptual understanding, reasoning skills, and problem-solving abilities (Bakker & van den Heuvel-Panhuizen, 2011; Van Galen & Van Eerde, 2018). However, the literature lacks comprehensive studies that examine its effects on the specific topic of multiplication of fractions. This research seeks to address this gap by conducting a rigorous investigation of the efficacy of RME in teaching this crucial operation.

The design of fraction multiplication learning activities within the context of RME was explored in this study. In designing and structuring the learning activities, we utilized Freudenthal's (1991) notion of guided reinvention as an element of the learning process. In an educational context, Treffers (1987) defined explicit mathematizing processes enriched by Streefland's (1991) observation and analysis. The main focus of this study is to explore how the mathematizing processes facilitate fifth-grade students' learning of fraction multiplication, as well as identifying the obstacles that hinder their learning. The following research questions guided this study:

1. How do mathematizing processes facilitate fifth-grade students' learning of multiplication of fractions using Realistic Mathematics Education-based instructional activities?
2. What obstacles do fifth-grade students encounter when learning multiplication of fractions using Realistic Mathematics Education-based instructional activities?

By answering these questions through an empirical investigation, this study aimed to contribute to the literature on mathematics education by providing insights into the effectiveness of RME in teaching multiplication of fractions and identifying potential areas for improvement in students' learning of the topic.

1.1. The Situation of Research and the Conceptual Model

The investigation was conducted at an international school in Ankara, Türkiye. We selected fifth graders because they learned multiplication of fractions during the semester in which the research was done. We extensively observed and analyzed the fractions learning process, especially when students were studying fraction multiplication. This study aimed to investigate critical elements of fraction multiplication learning and the mathematizing process that occurred throughout learning activities. The conceptual model that guided the research is visualized in Figure 1.1.

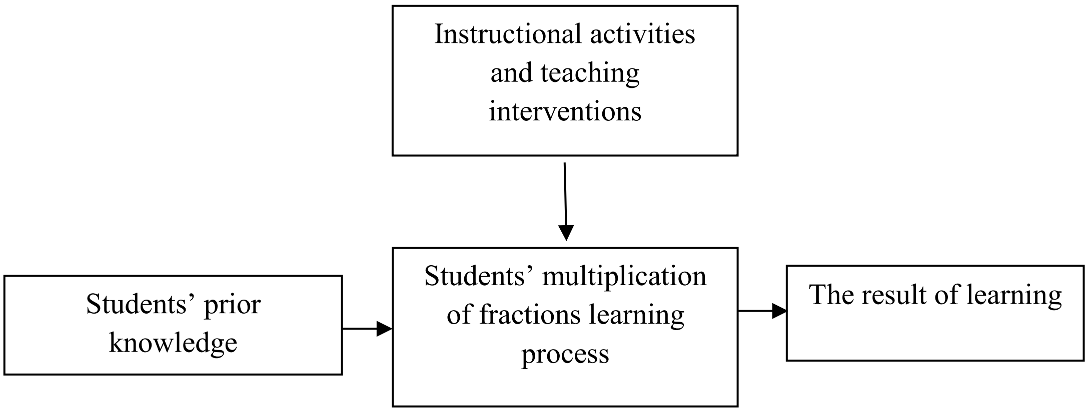


Figure 1.1. The conceptual model to guide the study

1.2. Significance of the Study

Working with fractions, particularly multiplication and division within a problem context or in ratio and proportion situations, is difficult for students of all ages. Lamon (2007) argues that fractions, ratios, and proportions are the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most crucial to success in higher mathematics and science.

Understanding fractions, along with solving problems in context (e.g., word problems) and mathematical comprehension, is commonly regarded as a critical aspect of student mathematics ability. According to Wu (2009), "learning fractions is the most critical step in comprehending rational numbers and preparing for algebra, as fractions represent students' first significant journeys into abstraction" (p. 8). According to Confrey et al. (2014), the scope of these challenges is as follows:

> Perhaps no idea in mathematics instruction is more crucial than rational number reasoning. Rational number reasoning, which is the foundation of the multiplicative ideas field, supports algebra, higher mathematical reasoning, and the quantitative competence required in science. Master indicates cognitive synthesis— comprehending, discriminating among, modeling, and integrating a

> remarkable variety of distinct yet closely connected concepts over a long period of time. (p. 968)

These challenges with fractions are in addition to the more general difficulty students experience in establishing linkages in word problems and algebraic equations (Keiran, 2007). When elementary, middle, and secondary preservice and in-service mathematics instructors were asked to model or provide representations-based solutions to fraction word problems, they struggled similarly, with many only able to provide procedural and symbolic solutions (e.g., Sjostrom et al., 2010; Olson & Olson, 2011). These findings raise concerns regarding the preparedness of teachers to address these issues with their students.

Despite the importance of mathematical content, the context in which that content is presented is essential. Although contextualizing mathematics is not a novel concept, the contextualization process has long been a topic of debate. Boaler (1993) asserted that "the specific context within which a mathematical problem is set is capable of impacting not only overall performance but also the selection of mathematical procedures" (p. 13). We identify our understanding of the significance of context in respect to the problem described in this study, which is a circumstance in which particular mathematical approaches, or in our case, representations and models, emerge as solution strategies. This study utilized four context problems: Running for Fun, Training for Next Year's Marathon, Exploring Playgrounds and Blacktop Areas, and Comparing the Cost of Blacktopping.

Although some research studies focus on developing learning activities to improve students' understanding of fractions, there is insufficient information about students' learning progress through the mathematizing process, as formulated by Treffers (1987) and now further developed by several researchers in RME, particularly in domain fractions, Keijzer (2003) and Streefland (1991). Keijzer (2003) proposed a different approach in curriculum sequence for learning fractions, in which mathematics teaching will no longer be based in separate learning strands, but will instead move from one meaningful situation with the opportunities to serve as a basis for more formal mathematics (vertical mathematization), rather than attempting to form more or less linear strands of activities primarily aimed at vertical mathematization. It begins with learning as a social enterprise, with the primary goal of creating conditions conducive to mathematics learning and encouraging students to explain how they construct their learning (Yackel & Cobb, 1996).

Multiplication of fractions is a complex mathematical operation that many students struggle to understand and apply due to the abstract nature of fractions. Traditional teaching methods often focus on repetitive exercises and procedural training, which may not facilitate deeper conceptual understanding of fractions (Putri & Dolk, 2017). Previous research has highlighted the effectiveness of the RME approach in promoting meaningful learning experiences and conceptual understanding of fractions (Khoiri, 2019; Lestari et al., 2019). However, there has been little research into the impact of using RME when teaching multiplication of fractions.

In an attempt to address the gaps in the research, this study studied how students acquire multiplication fractions through the mathematizing process. Modeling, symbolizing, generalizing, formalizing, and abstracting were all components of the procedure (Treffers, 1987; Streefland, 1991; Gravemeijer, 1994; Keijzer, 2003). This study investigated the feasibility and effectiveness of fraction multiplication activities designed for fifth graders. The designed learning activities were applied and enhanced to meet current educational requirements (Gravemeijer, 1994; cf. Gravemeijer, 2001). As a result, research was conducted to determine how students could engage in activities aiming to enhance their understanding of fraction multiplication.

The learning activities in the current study were constructed in the framework of RME, which means that students are guided to build on their informal knowledge by employing relevant and familiar contexts. In addition, these situations can lead to modeling, schematization, and therefore the formation of formal links between strands of numbers and other mathematical objects (Freudenthal, 1991; Gravemeijer, 1994; Van de Heuvel-Panhuizen, 1996). Starting with the RME paradigm, which requires the negotiation and construction of meaningful fractions, and therefore, classroom instruction might be considered interactive. Through the learning activities, students learned to relate meaning to fractions in various contexts, acquired a thorough understanding of fractions, and applied fractions in various situations (Greeno, 1991; Mcintosh et al., 1992).

The significance of this study lies in its investigation of the integration of RME approaches into the multiplication of fractions instruction for fifth-grade students. By exploring the application of RME strategies to multiplication of fractions, this study may bridge the gap in existing research and provide insights into how to improve the teaching and learning of multiplication of fractions.

Furthermore, this study has implications for pedagogical practices in mathematics education. By identifying effective RME approaches for teaching multiplication of fractions, educators can develop and integrate innovative teaching materials in mathematics instruction, ultimately promoting better achievement and deeper conceptual understanding among students.

This study speaks to the broader issue of promoting mathematical literacy and reducing achievement gaps in mathematics education. Research has shown that many students struggle with fractions and that gaps in mathematics proficiency exist across race and socio-economic status (National Center for Education Statistics, 2019). By investigating the impact of RME approaches on multiplication of fractions understanding, this study may help enhance teaching approaches that could improve mathematical literacy and close achievement gaps in mathematics education.

Another aspect of the significance of this study is the potential to provide insights into the role of visual representations in the multiplication of fractions. Previous research suggests that visual representations can facilitate conceptual understanding of fractions (Crespo, 2010; Fathurrohman & Wijaya, 2018). The RME approach uses contextual problems and visual representations to connect mathematical concepts to real-world situations, which can enhance students' understanding of mathematical concepts (van den Heuvel-Panhuizen & Dolk, 2012). This study will investigate the impact of using visual representations within the RME approach on students' understanding of multiplication of fractions.

Moreover, this study addressed the need for innovative mathematical teaching that meets the diverse needs of students. Many students struggle with mathematics, and traditional teaching methods have been said to contribute towards students' poor attitudes towards mathematics (Hadar et al., 2018). Engaging students through RME approaches, which utilize real-world situations and problem-solving strategies, could foster a more positive attitude towards mathematics. By investigating an innovative teaching approach within a specific mathematical concept, this study can contribute to a better understanding of how to develop effective pedagogical practices.

In conclusion, investigating the integration of RME strategies into the multiplication of fractions instruction for fifth-grade students is significant in advancing understanding of effective teaching practices, bridging gaps in research, and promoting mathematical literacy. This study could provide valuable insights into innovative approaches that enhance the teaching and learning of mathematics, which ultimately could improve students' attitudes towards the subject and their overall achievement in mathematics.

1.3. Theoretical Framework

A number of studies support the notion that students often encounter difficulty with the multiplication of fractions due to an inadequate understanding of the underlying concepts. For example, Gan, Potari, & Chua (2018) and Van Steenbrugge et al. (2014) note that many students mistakenly believe that multiplying two fractions results in a smaller fraction and may struggle with properly scaling factors up or down to achieve the correct result. Similarly, Bat-Cohen & Neria-Meir (2019) found that some students lack an understanding of the relationship between multiplication and equal-sized parts, making it difficult for them to perform accurate calculations involving fractions.

Other researchers have identified specific sub-concepts that play a role in students' misunderstandings of fraction multiplication. For example, Jacobs, Franke, & Carpenter (2007) highlight students' frequent confusion between multiplication and addition, leading them to fail to recognize equivalence relationships between different fractions. Similarly, Lamon (2012) argues that students require a deep understanding of the meaning of fractions as numbers in their own right, which involves recognizing that a fraction represents a part of a whole that can be scaled up or down to create equivalent fractions.

According to Putri & Dolk (2017), students may not comprehend the nature of multiplication and its basic properties, such as how fractions relate to natural numbers or how to account for scaling when multiplying fractions. Students may also struggle with the different representations of fractions, like their division representation as a ratio of two integers and the operator notation (e.g., $\frac{1}{2} \times \frac{2}{5}$ is $\frac{2}{10}$). Research has shown that lack of adequate conceptual understanding can impede deeper learning and prevent more meaningful connections between mathematics and the real world.

To address these challenges, a growing body of literature suggests that realistic mathematical modeling strategies can be valuable tools for supporting students' conceptualization and application of fraction multiplication within a Realistic Mathematics Education (RME) framework. Research has shown that such approaches can help students connect mathematical concepts to real-world contexts, thereby fostering a more meaningful understanding of mathematical relationships (Verschaffel, Greer, & De Corte, 2000).

Studies have found that realistic mathematical modeling can increase student engagement and motivation. For example, Koedinger, Alibali, & Nathan (2008) found that students were more motivated to solve math problems presented in real-world scenarios, which helped them build deeper conceptual understanding of mathematical relationships. Similarly, Moyer-Packenham, Westenskow, Hoyer, & Salkind (2014) found that realistic mathematical modeling activities helped to increase students' understanding of mathematical concepts.

The integration of realistic mathematical modeling and RME pedagogy has shown promising results in fostering deep learning experiences. Khoiri (2019) argues that realistic modeling can serve as a pedagogical opener for multiplication of fractions, allowing learners to explore the mathematical ideas through authentic contexts and hands-on activities. Results from previous studies such as Lestari et al. (2019) suggest that realistic contexts can offer opportunities for learners to utilize their prior knowledge, critical thinking, and problem-solving abilities.

Overall, these findings suggest that integrating realistic mathematical modeling approaches into an RME pedagogical framework can be a powerful tool for improving students' conceptualizations and applications of multiplication of fractions. With increasing concerns about math achievement gaps and limited resources for instruction, it is crucial to identify effective approaches that can promote meaningful learning experiences for students.

This study evaluated how students' understanding of fraction multiplication evolved when participating in RME-based educational activities. Using Treffers' (1987) framework, in which the mathematizing process served as a guide to observe and analyze learning progress, this study examined the development of students' understanding of fraction multiplication within the scope of the aim. According to Treffers (1987), there are two types of mathematization: horizontal and vertical mathematization. Horizontal mathematization is the process by which students use mathematical tools to solve problems encountered in daily life. In contrast, vertical mathematization is the process by which students reconstruct the mathematical system itself. Streefland (1991) has elaborated on the concept of the mathematizing process, which includes the modeling, symbolizing, generalizing, formalizing, and abstracting processes.

According to Gravemeijer (1994), modeling is the process of developing a situation-specific model, whereas symbolizing is the process of developing symbols to identify a suitable representation of the context (Linchevski, 1995). In addition, Linchevski (1995) stated that generalization is the process of developing mental constructs through pattern recognition. At this time, students will be aware of rules that apply to other objects. For example, the meaning of $\frac{1}{4}$ pizza is to divide into fourths and take one. Formalization is the application of rules, procedures, and algorithms to other mathematical problems (Hart, 1987). The final phase in the mathematizing process is the abstraction. During the stage of abstracting, students will link the higher and lower levels of thought (Streefland, 1991).

In this study, in addition to applying the mathematizing process paradigm, we developed a Hypothetical Learning Trajectory (HLT) to investigate students' learning of multiplication of fractions. According to Simon (1995), HLT (Figure 1.2) consists of three components: a learning objective/goal, a collection of learning activities, and a hypothesized of learning process. Teachers play a critical role in hypothesizing students' learning trajectories to provide activities linking students' current knowledge with their potential future thinking process. The produced HLT can be used for a single lesson, a series of lessons, or a learning concept over time (Simon, 1995).

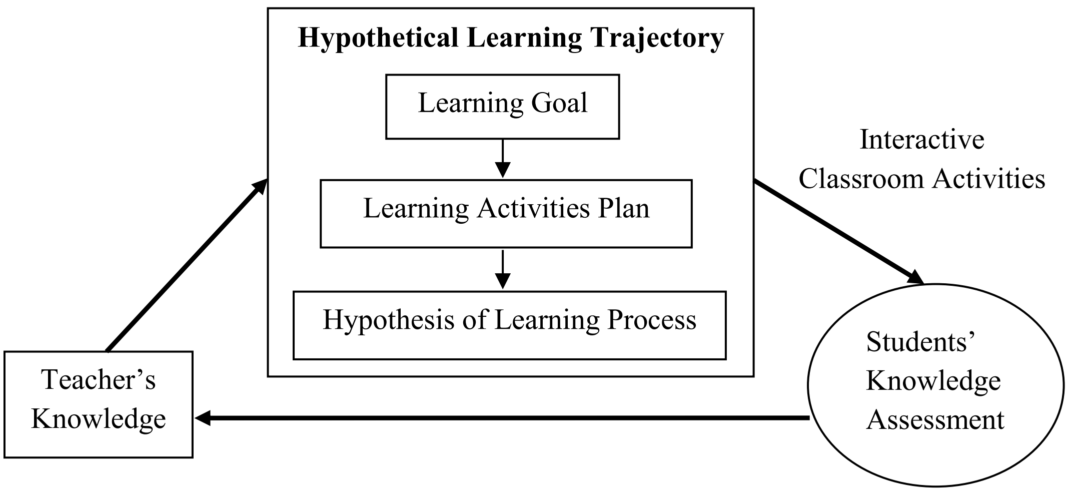


Figure 1.2. Hypothetical Learning Trajectory (Simon, 1995, p. 136)

1.4. Definition of Important Terms of the Study

In order to ensure clarity and specificity, this section defines the relevant terms and concepts that are important to the objectives of the study, both in constitutive and operational terms.

1.4.1. Realistic Mathematics Education

Realistic Mathematics Education (RME) is a domain-specific instruction theory that aims to provide effective mathematics education for students (De Lange, 1987; Treffers, 1987; Streefland, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996). According to RME, mathematics must be connected to reality, relevant to students, and close to their everyday experiences. In other words, if mathematical concepts and techniques are presented in a way that relates to the real world, they are more likely to be meaningful and useful for students.

One of the key features of RME is the use of realistic contexts in teaching mathematics. In RME, realistic contexts are considered essential for enabling students to learn and apply mathematical concepts and techniques to real-world problems. For instance, using a real-life scenario, such as calculating the area of a garden or the distance between two cities, can make a mathematical concept more meaningful and relevant to students.

Moreover, realistic contexts create a bridge between abstract mathematical concepts and the real world, which is particularly important for students who may struggle to see the relevance of mathematics in their daily lives. RME emphasizes the importance of connecting mathematics to students' interests and experiences, as this can foster motivation and engagement with the subject.

Another core features of RME is the use of models to represent and communicate mathematical ideas (Gravemeijer, 1999; Lesh & Zawojewski, 2007). Models are used to help students visualize mathematical ideas and understand how they relate to the real world. For instance, students who are learning about fractions may use models such as pies or rectangles to represent and manipulate fractions.

In addition, important feature of RME is the use of scaffolding to support students' learning (Van den Heuvel-Panhuizen, 2010). Scaffolding refers to the provision of support and guidance from a more knowledgeable other to help learners achieve a higher level of understanding and competence. In RME, teachers provide scaffolding through carefully designed tasks, questions, and feedback that help to guide students towards deep mathematical understanding.

Furthermore, RME emphasizes the importance of social interaction and collaboration as part of the learning process (Verschaffel et al., 2009; Doorman et al., 2012). In this approach, students are encouraged to work together to solve mathematical problems, discuss their reasoning, and justify their solutions. Through collaboration, students develop critical thinking, communication, and problem-solving skills.

In conclusion, RME is a theoretical framework for teaching mathematics that aims to make mathematics more meaningful and relevant to students. The use of realistic contexts is a key feature of RME and provides opportunities for students to learn and apply mathematical concepts and techniques to real-world problems. The use of models, scaffolding, and social interaction are key features of this approach, which are designed to support students' learning and understanding of mathematics. Overall, RME offers a promising framework for teaching mathematics that takes into account students' interests and experiences and helps to connect mathematical concepts to the real world.

1.4.2. Mathematizing

Mathematizing is a process in which students develop their own mathematical ideas through modeling, symbolizing, generalizing, formalizing, and abstracting (Streefland, 1991). In this process, students engage in exploring and understanding mathematics by creating mathematical models that represent real-world situations. By doing so, they discover mathematical concepts and procedures, and construct their own mathematical knowledge.

One of the key aspects of mathematizing is the use of modeling to represent mathematical ideas (Streefland, 1991). Models are used as a way of organizing and representing mathematical knowledge. For example, students may use manipulatives, diagrams, or visual representations to model a mathematical idea or concept. This encourages them to make sense of mathematical ideas by connecting them to real-world contexts.

Another important aspect of mathematizing is the process of symbolizing (Streefland, 1991). Symbolizing involves using mathematical symbols such as numbers, variables, and operators to represent mathematical ideas and relationships. Through symbolization, students learn how to communicate mathematically, and how to use symbols to represent and manipulate mathematical ideas.

In addition to modeling and symbolizing, mathematizing involves the process of generalizing (Streefland, 1991). Generalizing is the process of recognizing patterns and relationships, and using them to make general statements about mathematical concepts. By generalizing, students develop their ability to think abstractly and to make connections between different mathematical concepts.

Another aspect of mathematizing is the process of formalizing (Streefland, 1991). Formalizing involves transforming mathematical ideas and concepts into formal mathematical language and representations. This process enables students to communicate their mathematical understanding and to apply mathematical concepts and procedures to new situations.

Finally, mathematizing involves the process of abstracting (Streefland, 1991). Abstracting involves recognizing and using the essential mathematical structures and relationships underlying a particular problem or situation. By abstracting, students develop their ability to think critically and to apply mathematical concepts and procedures in new and unfamiliar contexts.

Mathematizing is a key instructional approach in RME that involves helping students develop the ability to mathematically represent, model, and analyze real-world situations. This approach is based on the belief that engaging students in authentic mathematical problem-solving tasks encourages deep understanding and transfer of mathematical concepts and skills to new situations (Gravemeijer, 1994). Through mathematizing, students learn to recognize, create, and connect mathematical ideas in the context of real-world problems and situations, and develop the skills and dispositions essential for productive mathematical thinking (Doorman et al., 2011).

According to Gravemeijer (1994), mathematizing is "the process of transforming informal and context-bound mathematical knowledge into more formal and general mathematical knowledge by looking at mathematical properties of phenomena and relating these properties to mathematical symbols and rules" (p. 28). Mathematizing, therefore, involves using mathematical reasoning and representations to make sense of real-world phenomena, and using these phenomena to develop mathematical concepts and skills. This process requires an iterative cycle of modeling, analyzing, and refining solutions that allows students to actively construct their own mathematical knowledge and understanding.

Research on the effectiveness of mathematizing as an instructional approach in mathematics education has shown promising results. For example, Doorman et al. (2011) found that students who participated in an RME program that included mathematizing tasks showed significant improvement in their ability to solve mathematical problems and transfer their knowledge to new contexts. Similarly, Muis and Mulder (2019) reported that mathematizing was an effective instructional approach for developing students' mathematical modeling skills.

In conclusion, mathematizing is a process in which students create their own mathematical ideas through modeling, symbolizing, generalizing, formalizing, and abstracting. This process enables students to develop a deep understanding of mathematical concepts by connecting them to real-world situations, recognizing patterns and relationships, and using symbols and formal language to communicate their ideas. By doing so, students become active participants in their own mathematical learning and develop the skills and knowledge they need to succeed in mathematics and beyond. Mathematizing is a key instructional approach in RME that promotes the development of students' mathematical reasoning and problem-solving skills by engaging them in authentic mathematical modeling and analysis of real-world problems and situations. Research suggests that mathematizing is an effective approach for developing students' mathematical knowledge and skills and strengthening their ability to transfer their learning to new contexts.

1.4.3. Guided Reinvention

Guided reinvention is a central design heuristic of RME that emphasizes the importance of allowing students to take ownership of their mathematical knowledge and thinking (Gravemeijer & Doorman, 1999). This approach is based on the belief that students' conceptions of mathematical concepts and methods must be taken into account when developing instructional strategies in mathematics education. Guided reinvention is intended to help students construct their own mathematical understanding and develop their ability to generalize, model, and apply mathematical concepts and methods to new situations (Freudenthal, 1973).

According to Gravemeijer (1994), guided reinvention involves presenting students with carefully designed tasks that enable them to gradually develop their own mathematical concepts and methods through exploration, reflection, and discussion. This process involves a series of steps, including identifying and analyzing the situation, looking for similarities and differences, formulating hypotheses, testing the hypotheses, and drawing conclusions based on the results. In this way, guided reinvention is an iterative process that encourages students to refine their understanding of mathematical concepts and methods over time.

Gravemeijer and Doorman (1999) argued that guided reinvention involves providing students with guidance and support as they engage in the process of reinventing mathematical ideas and concepts. This process involves a range of activities such as creating mathematical models, exploring relationships, and testing conjectures. Through this process, students build their own understanding of mathematical concepts, and learn to communicate their ideas effectively.

Additionally, guided reinvention is designed to enable students to view mathematical knowledge as their own private knowledge for which they are personally accountable (Gravemeijer and Doorman, 1999). This approach to teaching and learning emphasizes individual responsibility for understanding and developing mathematical ideas. By taking ownership of their mathematical knowledge, students become more engaged and personally invested in their learning, leading to improved retention and application of the concepts they have acquired.

Research suggests that guided reinvention is an effective approach to teaching mathematics. A study by Leung and Powell (2011) found that students who participated in an RME program that emphasized guided reinvention showed significant gains in their understanding of basic algebraic concepts. Similarly, a study by Bakker et al. (2014) found that RME instruction that focused on guided reinvention led to improved problem-solving skills and greater confidence in using algebraic concepts.

Research on the effectiveness of guided reinvention in mathematics education has shown that it is a powerful instructional strategy for promoting students' conceptual understanding and ability to generalize and apply mathematical concepts and methods (Mulligan et al., 2009). For example, studies have shown that students who are taught using guided reinvention tasks are more likely to transfer their mathematical understanding to new situations and to develop a greater appreciation for the usefulness of mathematics in real-world contexts (Ponce Campuzano et al., 2015).

In conclusion, guided reinvention is a central design heuristic of RME that emphasizes the importance of students taking ownership of their mathematical knowledge and thinking. By engaging in the process of reinventing mathematical concepts, students build their own understanding of mathematical ideas and learn to communicate their ideas effectively. It promotes students' ability to develop their own mathematical concepts and methods through exploration, reflection, and discussion. The research suggests that this approach for promoting students' conceptual understanding and ability to generalize and apply mathematical concepts and methods to new situations.

1.4.4. Hypothetical Learning Trajectory

The Hypothetical Learning Trajectory (HLT) is an instructional approach used in teaching and learning mathematical concepts and skills. It serves as a long-term strategy that guides educators in planning lessons by establishing learning goals, outlining activities, and predicting the hypothetical learning process over time (Simon, 1995). By following an HLT, teachers gain a clear direction and vision for designing learning activities and can envision how students will develop their understanding and reasoning abilities throughout the learning journey.

Smith et al. (2006) further elaborate that the HLT can be viewed as a predictive framework that outlines the expected progression of learners' thinking, understanding, and reasoning across various learning activities. It is crucial for the HLT to incorporate an overview of the mathematical concepts and skills to be taught, along with the specific strategies and activities that facilitate learning. Teachers who possess a comprehensive understanding of the learning trajectory can anticipate potential difficulties that students may encounter, enabling them to design targeted activities that help students overcome these challenges.

The HLT plays a crucial role in ensuring the continuity and coherence of learning by establishing a storyline for the development of a specific mathematical concept or skill over a designated period of time. Simon and Tzur (2004) emphasized that the HLT facilitates a clear sequence of learning activities that build upon students' pre-existing knowledge. By incorporating prior knowledge as a starting point, teachers can scaffold new information and skills upon this foundation, creating a cohesive and connected learning experience.

This approach allows educators to anticipate the challenges and difficulties that students may encounter during the learning process. With an HLT in place, teachers can promptly and specifically intervene to address these challenges. By identifying potential obstacles, educators can design targeted strategies, interventions, and instructional supports that help students overcome any difficulties they may face (Simon & Tzur, 2004).

Furthermore, Simon and Tzur (2004) argued that an effective HLT should align with learners' prior knowledge and be tailored to their specific cognitive development. By recognizing and considering the individual differences in students' backgrounds and cognitive abilities, teachers can ensure that the learning trajectory is appropriately designed to optimize learning outcomes.

Another crucial aspect of the HLT is its focus on the development of learners' reasoning and problem-solving skills. The HLT serves as a framework through which teachers can design learning activities that promote deep learning and encourage learners to engage in mathematical reasoning and critical thinking (Remillard, 2005).

By following the HLT, teachers can create activities that go beyond rote memorization and surface-level understanding. Instead, these activities challenge learners to apply their knowledge and skills in novel situations, fostering the development of higher-order thinking skills. Through problem-solving tasks and open-ended questions, learners are encouraged to analyze and evaluate mathematical concepts, make connections between different ideas, and generate their own mathematical strategies and solutions (Remillard, 2005).

Additionally, the HLT allows teachers to consider the progression of learners' thinking and reasoning abilities as they move through the trajectory. This understanding helps teachers design activities that build upon and extend learners' current capabilities, providing appropriate scaffolding and challenges to support their growth (Remillard, 2005). By providing opportunities for learners to grapple with complex problems, make conjectures, and justify their reasoning, the HLT facilitates the development of robust mathematical reasoning skills.

There is a substantial body of research on the HLT and its impact on mathematics education. One study by Remillard and Bryans (2004) focused on how teachers developed and utilized HLTs to design learning activities that fostered the development of mathematical thinking and understanding. The findings revealed that teachers effectively employed the HLT to anticipate possible difficulties that students might encounter along the learning trajectory. By actively considering these challenges, teachers were able to design activities that facilitated the connection of new knowledge with students' prior knowledge.

Similarly, Walkoe and Moschkovich (2009) conducted a study that examined the use of an HLT by teachers to support the development of learners' understanding of algebraic reasoning in a middle school context. The study demonstrated the HLT's efficacy in designing learning activities that aided learners in comprehending algebraic thinking. The teachers utilized the HLT as a guide in creating a series of progressively challenging and interconnected activities. This sequential approach allowed learners to develop their thinking skills incrementally over time.

These studies provide empirical evidence for the effectiveness of the HLT in supporting teachers' instructional design and students' learning. By utilizing the HLT, teachers are empowered to anticipate and address learning challenges while creating a coherent and connected learning experience for their students. Furthermore, the HLT serves as a valuable framework for fostering the development of learners' mathematical thinking and understanding.

In conclusion, the HLT has emerged as a powerful tool for teachers in designing learning activities that promote students' mathematical thinking and understanding.

The HLT provides a roadmap for teachers, allowing them to anticipate potential difficulties that learners might encounter along the learning trajectory. By actively considering these obstacles, teachers can design activities that engage students and help them connect new knowledge with their prior understanding.

Furthermore, the use of HLTs has been shown to promote deep learning and enhance students' performance on mathematical tasks. This approach enables teachers to create progressive and interconnected learning activities that build upon each other. As learners engage with these activities over time, they are able to develop their thinking skills and gain a deeper understanding of mathematical concepts and skills.

The HLT offers teachers a valuable approach to instructional planning, as it empowers them to tailor learning experiences that align with learners' needs and promote mathematical thinking and understanding. By utilizing the HLT, teachers can enhance their instructional practices and create meaningful learning experiences for students.

CHAPTER 2

LITERATURE REVIEW

This chapter presents the conceptual underpinning for this study. To construct the instructional activities, the literature on fractions, big ideas in mathematics, big ideas and strategies underlying multiplication with fractions, and the types of fraction models were investigated. We also explore Realistic Mathematics Education (RME), the mathematizing process, and the principles of RME teaching and learning to explain and investigate how the framework of studying fractions introduces students to more formal mathematics. In addition, as the study was undertaken in an American curriculum-based school, this chapter includes a summary of the Common Core State Standard Initiative regarding fraction learning in fifth-grade.

2.1. Fractions are Numbers

Number sense is complex. Nevertheless, making sense of numbers continues to be a priority for many educators and students. On the other hand, the more aspects of number sense are considered, the more difficult it is to describe, discuss, and evaluate it precisely. The development of a sense of number in relation to fractions (identified here as $\frac{a}{b}$ fractions, decimals, and percentages), also known as fraction sense, is common and important for students in their middle school years. This is because work with fractions actually begins early on and expands students' perceptions of algebra.

In its Curriculum and Evaluation Standards for School Mathematics, NCTM describes number sense as an intuition about numbers derived from the many meanings of number (NCTM, 2000). This includes grasping the meanings of numbers, evaluating different relationships between numbers, identifying the relative magnitude of numbers, comprehending the influence factors of operations on numbers, and producing referents for measures of common items and situations.

Some research found that developing a sense of natural numbers is easier than developing a sense of numbers with fractions. For instance, according to NCTM (2007), fifty percent of eighth graders were unable to arrange the three fractions in the correct order, from smallest to largest. Kloosterman (2010) found that fewer than thirty percent of seventeen-year-olds were able to express 0.029 correctly as $\frac{29}{1000}$ of the total. While Rittle-Johnson et al. (2001) found that when the students were asked which of two decimals, 0.274 and 0.83, is greater, they mistakenly selected 0.274 as the answer.

Sowder and Schappelle (1994, p. 342) observed that "number size is not a perception about numbers that once obtained for natural numbers extends to all sorts of numbers." To restate, students can have a sense of numbers when they work with natural numbers, but not when they work with fractions. Yet, this is not all. In its final report, the NMAP (2008, p.18) made the observation that "proficiency with fractions (including decimals, percent, and negative fractions) appears to be the most critical core skill not yet established" among the foundational abilities necessary for algebra.

The article Creating Effective Fractions Instruction for Kindergarten through Eighth Grade (Siegler et al., 2010) makes several recommendations, one of which focuses on helping children grasp that fractions are numbers and that they expand the number system beyond natural numbers. This advice is compatible with the emphasis on fractions, ratios, and proportional relationships in the Common Core State Standards for Mathematics (CCSSM) (NCTM, 2022).

Fractions are an integral part of the number system and are considered essential in mathematics education. Fractions represent parts of a whole and are used to describe quantities that are not natural numbers (Lamon, 2012). Fractions are also used in real-life situations, such as in cooking recipes, measurements, and financial calculations. It is important for students to understand the concept of fractions as numbers to develop strong mathematical skills (Behr et al., 1992).

One of the key concepts in understanding fractions as numbers is the idea of equivalence. Fractions can be equivalent, which means that they represent the same amount or part of a whole. For example, $\frac{1}{2}$ is equivalent to $\frac{2}{4}$ and $\frac{3}{6}$. This concept is important in order to compare, add, subtract, and multiply fractions (Lamon, 2012).

Another important concept in understanding fractions is their relationship to decimals and percentages. Fractions can be expressed as decimals, which are based on a base-ten system and are used in calculations involving money and measurement. For example, $\frac{1}{2}$ can be expressed as 0.5, and $\frac{3}{4}$ can be expressed as 0.75. Fractions can also be expressed as percentages, which are based on a base-100 system and are used to represent parts of a whole. For example, $\frac{1}{2}$ can be expressed as 50%, and $\frac{3}{4}$ can be expressed as 75% (McGehee, 2019).

It is important to recognize that fractions have unique properties that distinguish them from other numbers. Fractions can be neither negative nor positive, and they can be added and subtracted only if they have the same denominator. Understanding these properties is important in developing students' mathematical reasoning skills (Behr et al., 1992).

Fractions are fundamental to the study of mathematics as they reflect the necessary transition from natural numbers to numbers that partition quantities or entities that are not logically divisible along simple base ten lines. By breaking down wholes into equal parts or pieces, fractions provide a way to articulate the relationship between one and the parts that make up a whole (Borum, 2017). Additionally, understanding the connection between fractions, decimals, and percentages enables students to make correct calculations in real-world applications and solve complex problems with ease (Faulkenberry, 2021).

A major challenge for students when learning fractions is understanding the correlation between the way fractions are visualized and the numeric concepts being conveyed. Understanding the rule of fraction arithmetic is fundamental for students to reason mathematically and solve fraction problems accurately. Furthermore, it allows students to feel confident using fractions as numbers because they understand what it means to add and subtract them (Tournaki & Bae, 2018).

Recognizing and interpreting the properties of fractions is also a key component in comprehending the relationship between fractions and numbers. Students need to understand the unique properties of fractions, such as the inverse relationship between the numerator and denominator (i.e., the larger the numerator, the smaller the fraction), applying rules of fraction arithmetic, and understanding the relationship between fractions, decimals, and percentages (McDermott & the Tensor Development Team, 2019).

In many fraction instructions, the concept of part of the whole is frequently emphasized (Siegler et al., 2011). One-third, for instance, is one portion of a total divided into three pieces. Siegler claimed that the concept of a portion of a whole is crucial but cannot transmit the concept of fractions as magnitude. For instance, fractions such as $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \cdots$ can be equivalent or arranged from smallest to largest or vice versa. According to Charalambous et al. (2010), students who exclusively learn fractions as a part of a whole typically struggle with fractions higher than one. Students might conclude, for instance, that $\frac{5}{4}$ is not a number since they cannot create five parts from four parts.

There is agreement among researchers regarding fractions interpretations (Marshall, 1993; Petit, Laird & Marsden, 2010; Steffe & Olive, 2010; Clark & Roche, 2011; Empson et al., 2011), namely: a linear interpretation (e.g., $\frac{3}{4}$ represented on a number line), a part-whole interpretation (continuous or discrete model), a part-part relationship, a quotient (or division) statement, and an operator (e.g., $\frac{3}{2}$ of a recipe).

Sowder and Schappelle (1994) argued that students must notice and comprehend that fractions are numbers themselves as a fundamental component of fraction sense. Consider the following numerical analogy. Students in the early grades interpret the number 42 as 4 tens and 2 ones, or 40 + 2, and depict such numbers using various physical and, more recently, digital methods. As students work with addition and subtraction, they compare, order, and generally comprehend natural numbers. However, many students interpret $\frac{9}{10}$ as 9 over 10 or as part of a whole, which may be the case, but $\frac{9}{10}$ is also a number. It is close to 1 and can be expressed by the fractions 0.9, 90%, $\frac{18}{20}$, and other equivalent fractions.

According to Hannula (2003), while students may be proficient with fractions, many do not appear to really realize that fractions are numbers. He emphasized the importance of students understanding fractions as an extension of the number system. His research detailed some significant issues confronting children aged 12 to 14. Many of these challenges are thought to develop because students see fractions as portions of a shape or amount rather than numbers. Only the part-whole model was recognized by all the students who participated in his research. He thought the issue started in elementary school when fractions were initially given as geometric picture components. He claimed that children are not provided enough cues in the classroom to understand that fractions are numbers. In work involving graphs, algebraic equations, and numerical patterns, integers are often used solely.

Students had difficulty identifying the unit in part-whole diagrams that depict more than one unit, according to research that was conducted by Domoney (2002). A significant number of students will respond $\frac{7}{10}$ rather than $\frac{7}{5}$ when a graphic is used to illustrate a fraction that is bigger than one, such as the one shown in Figure 2.1. When separate part-whole diagrams are used to demonstrate the addition of two proper fractions (see Figure 2.2) or when the sum is higher than one-unit, similar complications arise (see Figure 2.3).

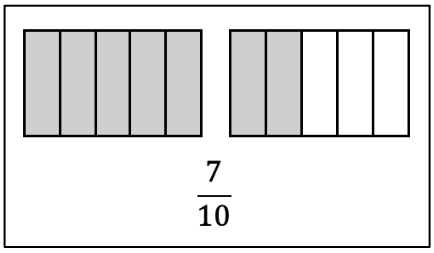


Figure 2.1. A representation of $\frac{7}{10}$

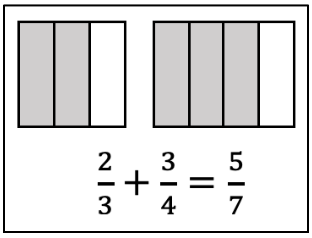


Figure 2.2. A representation of $\frac{2}{3}+\frac{3}{4}=\frac{5}{7}$

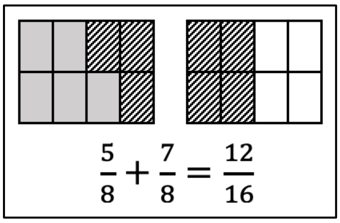


Figure 2.3. A representation of $\frac{5}{8}+\frac{7}{8}=\frac{12}{16}$

According to Behr et al. (1992), some misunderstandings could result from new ideas not being strongly associated with the student's prior ideas. On the other hand, some additional misunderstandings could result from the absence of some essential aspect of the knowledge scheme which has been ignored in the design of the teaching material. Consequently, certain misunderstandings may also be related to instructional constraints, which may result in students constructing a schema in a more constrained manner. This may be the cause of certain misconceptions. The practice of giving names to improper fractions, as shown in Figure 2.1, or adding the numerators and denominators in the process of adding fractions, as shown in Figures 2.2 and 2.3, may be the outcome of using a more constrained scheme for fractions. It is indeed possible that the student will think of fractions as nothing more than a pair of natural numbers with one printed on top of the other. Students need to be able to distinguish between natural numbers and fractions and integrate these two types of numbers before they can conceptualize rational numbers. Therefore, the lack of incorporation does not necessarily suggest a flexible representational model. It seems vital to utilize numerous models for each concept; nevertheless, where it is practicable, two or more related concepts ought to be represented together for the relationship between them to become evident. One illustration that pertains to this investigation is the practice of employing multiple representations to work simultaneously with natural numbers and fractions to bring attention to the connections between the two different kinds of numbers.

Hodges (2007) discovered that the use of diagrams was occasionally helpful in the resolution of problems involving fractions, or that they were used to assess whether the answer provided was possible. Yet, comprehending a part-whole picture typically entailed the steps including: (i) counting the number of pieces that had shading applied to them; (ii) counting the total number of pieces; and (iii) finally, writing one complete number on top of the other. During the interviews, right after the students answered the question about the fraction shaded in a part-whole diagram for $\frac{3}{5}$, the next question that was asked of them was the fraction that was not shaded. According to Hodges's findings, very few students deducted the fraction shaded from one $(1-\frac{3}{5})$; instead, they frequently employed the counting method that was just explained. It is possible to speculate at this point that those students correctly identified the fractions without being aware of the relationship between the fraction $\frac{5}{5}$ and the natural number 1. In point of fact, the process of counting and labeling a fraction does not need the application of any concept of fractions as parts of a whole. Instead, it just involves counting. This fraction representation can be understood as a pair of natural numbers.

In addition, research has shown that students have difficulty recognizing a correct fraction when it is shown on a number line with two units of length rather than one unit of length (Hannula, 2003). There is a widespread misunderstanding that the fraction $\frac{1}{n}$ should be placed at the $(\frac{1}{n})$th of the distance from 0 to 2. So, it would appear that some students have difficulty identifying the unit when using number lines, much like they do when using part-whole diagrams.

Pirie and Kieren (1994) describe how 10-year-old Katia attained "a new understanding" (p. 174) of the addition of unrelated fractions (halves and thirds) by drawing part-whole diagrams (pizzas) for the fractions and then dividing both into sixths. This was done despite the fact that part-whole diagrams are thought to be misleading and a possible inhibitor of the development of other interpretations for fractions. There is also a degree of consensus that fractions ought to be taught as individual components of a whole (English & Halford, 1995). Possibly due to the fact that it is the first feature of fractions that a student encounters in their lifetime. So, additional investigation into the means by which a transition from the part-whole perspective to the perspective of fractions as numbers could be accomplished is required.

The psychological theory of mathematics education that English and Halford (1995) created integrates psychological principles with theories of curriculum development. This theory was produced by the two researchers who emphasized how important it is for students to construct their mathematical knowledge using their prior knowledge, and how important representations and analogical reasoning are. Nonetheless, the choice of representation and the operations that are carried out on it might have significant repercussions for the student's progress in mathematical study. Such representations can make the topics they are meant to teach students more difficult to understand or distort them. Some representations, such as imaginary stories such as mating occurs exclusively between fractions, therefore mixed numbers, $1\frac{2}{3}$, become improper fractions, $\frac{5}{3}$, may help students remember methods, but they do little to create a mental knowledge of the subject matter (Chapin, 1998). The literature (English &

Halford, 1995) suggests significant pedagogical and physical criteria should be used to select representations.

Skemp (1986) suggests that educators select representations that are flexible and can be utilized in the process of constructing long-term schemas. A short-term schema, which will likely need to be reconstructed in the near future, is easier to assimilate than one that is applicable to a large number of mathematical concepts and therefore makes the assimilation of later concepts easier. The criterion for choosing representations that can be used in a variety of contexts is what English and Halford (1995) refer to as the principle of scope. They believe that the part-whole model is a representation that has scope since it may show many different concepts and procedures relating to fractions.

Fractions are a crucial component of understanding mathematical concepts, especially as students advance to higher-level math courses. When students understand that fractions are numbers, they develop a deep conceptual understanding that will allow them to confidently and accurately apply fraction operations to their work. For example, understanding that adding a fraction to a natural number involves the same fundamental principles as adding two natural numbers, allows students to understand that fractions are real, meaningful numbers that are part of the broader number system (Van de Walle et al., 2017).

One challenge in teaching fractions as numbers is the tendency for students to think of fractions as ratios instead of numbers. They may not realize that fractions can be represented on the number line and can be compared using the same methods as integers. Featuring a visual representation of fractions, such as a pie or a bar model, enables students to see fractions as numbers that can be aligned with other numbers on the number line. This approach helps students make connections between different representations of numbers, building a solid foundation for future math concepts in higher-level mathematics classes (Battista, 2018).

Moreover, teachers need to present fractions in an integrated, contextualized manner rather than as disconnected operations to ensure students are grasping the concept as numbers. They may use real-life examples such as baking recipes or construction blueprints to teach fractions as numbers. This teaching approach helps students connect fraction concepts to their everyday lives and prepares them to apply their knowledge of fractions logically and efficiently beyond the classroom (Hoffer et al., 2020).

In conclusion, treating fractions as numbers is vital for proficiency in mathematics. Teaching fractions in context, providing opportunities for visual modeling, and introducing connections to other operations within the number system are all important steps in fostering a deep conceptual understanding of fractions. In doing so, students will be better prepared to apply the reasoning and logic skills required for success in higher-level math courses.

2.2. Big Ideas in Mathematics

Mathematics extends far beyond arithmetic operations and problem-solving techniques; it involves big, overarching concepts that establish a strong foundation for mathematical reasoning. These big ideas serve as guides for teachers and students, promoting deeper understanding and facilitating connections across different mathematical topics. The following are a few examples of big ideas in mathematics and some references that provide additional information:

1. *Patterns*. Patterns are evident in nature, art, music, and many other aspects of daily life. In mathematics, patterns are incredibly helpful for predicting outcomes, identifying similarities, and analyzing changes. Encouraging students to recognize patterns in numbers, shapes, data, and processes helps them to develop a more profound understanding of mathematics and its potential applications (NCTM, 2014).

2. *Connections*. Mathematics is an interwoven subject, and different areas are more connected than students might expect. For example, understanding geometry is vital for studying algebra, and fractions are a fundamental component of

understanding rational functions. Building connections between different topics can illuminate new opportunities for solving problems and enable students to see the relation of these concepts to the real world (Boaler, 2019).

3. *Representations*. Presenting mathematical concepts through visual representations, tables, graphs, and symbols is crucial for deep comprehension. Math concepts can be abstract, but using a variety of representations can help students grasp the meaning better. For instance, graphing functions enables students to see data patterns and understand different types of functions better (National Center on Intensive Intervention, 2018).

4. *Logic and Reasoning*. Mathematics is about logical reasoning, and beginning to understand the concept and apply it to problem-solving early on is essential. Emphasizing that mathematics is logical can help students address complex problems more effectively (National Council of Supervisors of Mathematics, n.d.).

5. *Proportional Reasoning*. Proportional reasoning is an essential big idea because it connects key mathematical ideas from a wide range of topics, such as measurement, numbers, algebra, and geometry. Proportional reasoning involves understanding how two quantities are connected in terms of multiplication. It is crucial to recognize that proportional relationships are not limited to just two aspects, but two or more quantities can be proportional. Developing proportional reasoning is essential as it helps students to compare quantities and solve complex mathematical problems (NCTM, 2014).

6. *Mathematical Modeling*. Mathematical modeling is the process of using mathematics to develop a representation of a real-world situation. It involves problem-solving, critical thinking, and decision making which will help students to apply mathematics in real-world situations. By requiring learners to apply mathematical modeling to various scenarios, they will learn how mathematical concepts and formulas function in the actual world (Lesh & Doerr, 2003).

7. *Number Sense*. Number sense is a fundamental big idea in mathematics and is the ability to comprehend the relationship between numbers and their values. In other words, it's about understanding the meaning and structure of numbers and their properties. According to research, number sense is a critical factor in developing computational skills, grasping higher-level mathematical concepts, and solving related problems (National Research Council, 2001).

8. *Algebraic Thinking*. Algebraic thinking is another significant big idea in mathematics that emphasizes the use of symbols or letters to represent numerical relationships. It involves analyzing and solving problems using patterns, symbols, and a deeper understanding of mathematical concepts. Algebraic thinking is also essential because it builds a foundation for higher-level mathematical concepts, such as calculus and abstract algebra (NCTM, 2014).

9. *Spatial Reaso*ning. Spatial reasoning is a crucial big idea because it plays a role in many aspects of mathematics and science, including geometry, measurement, and even statistical data visualization. Spatial reasoning involves identifying objects, understanding spatial relationships, and being able to transform shapes, a skill that is important for mathematical problem-solving (National Research Council, 2006).

10. *Mathematical Practices*. The NCTM has identified eight essential mathematical practices that can serve as a guide to incorporating big ideas into math instruction. These practices include problem-solving, reasoning, modeling, using tools and technology, communication, connections, representations, and visualization. These practices can enhance the learning experience in mathematics and support students' ability to apply math concepts in real-life situations (NCTM, 2014).

There is nothing novel about the concept of teaching mathematics using big ideas. Charles (2005) observed that information acquired without appropriate structure is the knowledge that is prone to be lost. A declaration of an idea that is important to the learning of mathematics and links various mathematical understandings into a cohesive whole, is what Charles (2005) meant when he referred to a big idea in mathematics. For instance, most people agree that place value is a big idea for this initiative (e.g., Charles, 2005; Siemon, 2006; Van de Walle et al., 2010; Askew, 2013). However, beyond this point, there is very little consensus regarding these big ideas or how they should be represented to best support the teaching and learning of school mathematics. According to Askew (2013) and Siemon et al. (2012), big ideas need to have both a significant impact on mathematics and an acceptable place in the classroom.

The big ideas in mathematics provide mathematics educators with an organizing framework within which they can think about the issue at hand in the context of their role as mathematics educators. When teachers are aware of these notions and their position in the mathematical landscape, they are able to look backward and organize their lessons based on a knowledge of where their students are in the mathematical landscape. This allows teachers to adapt their instruction to meet the needs of their students. They are also able to predict the mathematical futures of students and the possibilities presented by the insights offered by students as they struggle with the mathematics of the present. This ability allows them to foresee the mathematical futures of students. In other words, they are able to foresee the mathematical paths that students will take in the future.

Many attempts have been made to characterize what it is that makes up a big idea to develop a profound and well-connected understanding of mathematics (for example, Schweiger, 2006; Siemon, 2008; Kuntze et al., 2011; Askew, 2013). Yet, even though these initiatives have inspired many possible Big Ideas, they have not led to a consensus over the characteristics of what constitutes a big idea for this initiative.

Some authors have come to a conclusion, based on the fact that there is a lot of variation in the literature on big ideas, that we may never arrive at a shared understanding of what it means to have a big idea in order to support a coherent approach to the teaching and learning of mathematics in schools (e.g., Charles, 2005;

Clarke et al., 2012). To shed some light on which big ideas might be more valuable than others in building a comprehensive understanding of mathematics over time, however, a closer look is necessary as it may shed some light on which big ideas are currently in use. The following are the two aspects of big ideas variation, namely size of big idea and mode of expression.

Size of big idea. The subject matter of mathematics taught in schools has, from the beginning, been subject to classification on a broad scale (e.g., number, measurement, chance). The categories shift over time as a result of shifts in the academic field (for example, the new mathematics of the 1980s) and shifts in the expectations of society (e.g., the Programme for International Student Assessment (PISA) 2022 Mathematics Framework). Yet, despite the fact that these categories can be helpful in defining essential mathematical processes and showing linkages. In order to improve instruction and better facilitate student learning, it is necessary to develop a more refined set of big ideas and strategies, as well as the relationships that exist between them (Charles, 2005; Siemon et al., 2012; Askew, 2013; Tout & Spithill, 2015).

Mode of expression. The concern here is that some big ideas, such as function and infinity, are expressed as concepts, while others are expressed as processes, such as explaining and structuring. According to Askew (2013), this presents a challenge regarding its consequences on teaching. It seems that the many ways of expressing ideas are connected to identifying big ideas. Examining the mathematical structure is the foundation of one methodology, which Hiebert and Carpenter (1992) call a top-down approach. The big ideas uncovered in this manner typically take the form of declarative assertions, such as the following: "fractions, decimals, and percentages all present alternative ways to depict a multiplicative relationship between two quantities" (Askew, 2013, p. 8).

An additional method for determining big ideas, which is analogous to the bottom-up approach proposed by Hiebert and Carpenter (1992), is based on investigating and documenting the increasingly sophisticated understandings constructed by students over time as a result of their participation in mathematical activity. When seen from this angle, the big ideas tend to be described in terms of evidence-based descriptions of major mathematical ideas and processes over time. For instance, equipartitioning (Confrey et al., 2014) and multiplicative thinking are two examples of these types of descriptions (Siemon et al., 2006, 2012).

Big ideas based on a top-down approach. From this point of view, the interpretation of a big idea offered by Charles (2005) is the one that is most frequently referenced. The term "big idea" refers to a summary of an understanding fundamental to the study of mathematics and that brings together a variety of mathematical concepts into a unified whole (p. 10). In his book, Charles identifies twenty-one big ideas in mathematics and provides examples of mathematical understandings for each. For instance, for a big idea on base ten numeration system, the example of mathematical understandings related to natural numbers includes: (i) Objects, phrases, and symbols can all stand in for numbers; (ii) A digit's location in a number indicates how many ones, tens, hundreds, etc. it stands for; (iii) Each digit to the left of a given digit is equal to ten digits to the right (e.g., $100 = 10 \times 10$); (iv) The value of the number can be calculated by adding together its component digits; and (v) To correctly interpret numbers using place value, groups of ten, a hundred, etc., must be viewed as single entities.

Big ideas based on a bottom-up approach. When seen from this perspective, big ideas tend to be associated with evidence-based interpretations of important mathematical processes across time (e.g., unitizing, relational thinking: equipartitioning). Although they are deeply connected to the structures of mathematics, the big ideas identified by the bottom-up approach are considered big for two reasons: first, because they are essential to the field of mathematics, and second, because they represent significant advances in the growth of student's reasoning (Fosnot & Dolk, 2002).

The study on learning progressions and trajectories is a good example of evidence-based approaches that may be used for the goal of finding the big ideas for teaching and learning (e.g., Clements & Sarama, 2009; Siemon et al., 2006, 2019; Confrey et al., 2014, 2017).

One publication that provides an illustration of this method's big ideas is Confrey et al. (2017) with its four fields and nine big ideas of the learning map" as seen in Figure 2.4 below.

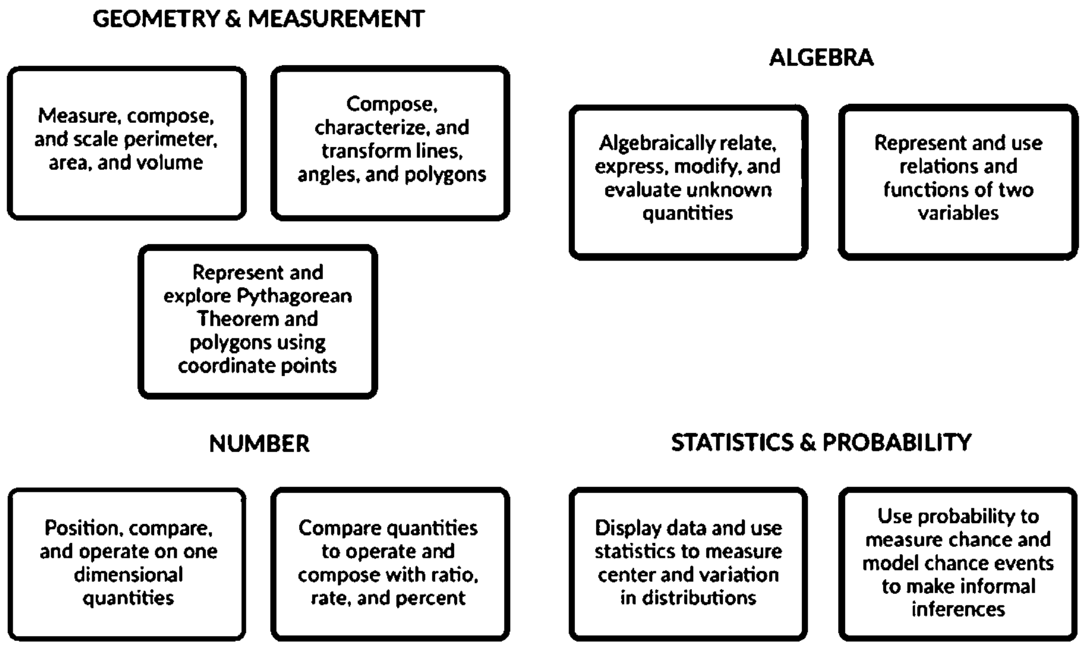


Figure 2.4. Four fields and nine big ideas of the learning map (Confrey et al., 2017, p. 721)

In conclusion, big ideas in mathematics play an integral role in building a foundation for mathematical understanding. It is essential for teachers to incorporate these concepts to create curricula and individual lesson planning. Teachers must incorporate these concepts into their classroom instruction and allow learners to explore them more through real-life problem-solving activities and relating the mathematical concepts to everyday life situations. It is also necessary for students to grasp these concepts early on and get a hold of them throughout their math journey for mastering mathematics concepts. It is important to recognize that big ideas, such as proportional reasoning, mathematical modeling, number sense, algebraic thinking, spatial reasoning, and mathematical practices, can engage student interest and promote curiosity while fostering mathematical proficiency.

2.3. Big Ideas Underlying Multiplication with Fractions

As described in the previous sub chapter, big ideas are also related with transformations in students' reasoning, including shifts in perspective, logic, and mathematical relationships. According to Hellman and Fosnot (2007), fraction multiplication is built on four big ideas:

2.3.1. Fractions Represent a Relation

Fractions represent a relationship between two quantities. They describe a part of a whole or a portion of a quantity, and they are essential to multiple mathematical concepts, making a deep understanding crucial. One important big idea underlying the concept of fractions is that fractions represent part-whole relationships. Understanding this relationship is crucial to build upon and connect fraction sense to other mathematical concepts. Fractions represent a relationship between two quantities. They describe a part of a whole or a portion of a quantity, and they are essential to multiple mathematical concepts, making a deep understanding crucial. One important big idea underlying the concept of fractions is that fractions represent part-whole relationships. Understanding this relationship is crucial to build upon and connect fraction sense to other mathematical concepts.

A part-whole relationship can be represented using a variety of models. One of the most common and useful visual models is area models. An area model breaks down a shape into equal parts, with each fraction corresponding to a different part of the whole. Students can use area models to see how fractions are represented in a visual and identify the relationship between the fractional part and the whole. The fractional parts can aid students' calculations and provides an intuitive understanding of fractions' meaning that can be used when solving problems. (National Research Council, 2006)

Another big idea underlying the concept of fractions is that fractions can be used to represent ratios and proportions. Ratios are a comparison between two quantities, and proportions compare two ratios. Fractions can represent both ratios and proportions; a ratio of 1 to 3 can be expressed as $\frac{1}{3}$, or a proportion of 2:3 can be expressed as $\frac{2}{3}$. This big idea is important for understanding fractions in real-world contexts, where fractions can be used to represent a part of a whole or a percentage of a larger quantity.

It is crucial that, to develop a comprehensive understanding of fractions, students must grasp the idea that fractions are a mathematical representation of a part-whole relationship and can be used to represent ratios and proportions. This understanding lays the foundation for more complex mathematical concepts that rely on fractions, including decimals, percents, and algebraic equations.

In conclusion, fractions represent a relationship between two quantities and are essential to multiple mathematical concepts. By understanding the part-whole relationship and how fractions can represent ratios and proportions, students can gain a deeper understanding of fractions necessary to master more complex mathematical concepts.

2.3.2. The Whole Matters

One big idea that underlies the concept of fractions is that the whole matters. This idea suggests that understanding the relationship between the whole and its fractional parts is essential to develop a deep understanding of fractions. It is crucial for students to understand the significance of the whole when working with fractional parts as a basis for problem-solving. Without understanding the whole, students may not comprehend the context and significance of the fractional parts (Parrish & Wright, 2010).

One way to promote this big idea is the use of real-world contexts and problem-solving tasks. Students can work with situations that involve fractions representing parts of a whole. They can break down an object or a set of objects into parts and work with the parts as fractions to find the relationships between the parts and the whole. These tasks can help students understand how fractions can represent parts of a larger quantity of objects or a portion of an area (NCTM, 2014).

Another important aspect that contributes to realizing the big idea of fractions as a whole is the use of visual models. A variety of models can demonstrate fractions as part of a whole, including number lines, fraction bars, and area models. These models demonstrate the relationships between fractions and the whole in a visual way, allowing students to see the significance of the whole as a basis for fractions. Using visual models can strengthen conceptual understanding by providing concrete representations of fractional parts (NCTM, 2014).

According to Van de Walle et al. (2013), the whole and its unit form the basis of fractional thinking and comprehension. Students need to recognize that a fraction represents part of a whole, and solving fraction problems requires identifying the relationship between the fractional parts and the whole. Teachers can encourage students to develop this understanding through problem-solving tasks that use the concept of part-whole relationships. For example, pizza can be divided into equal parts and given to specific numbers of students to highlight the relationship between the whole (the pizza) and the fractional parts (the slices). This can help students understand the concept of fractions beyond isolated, numerical calculations. By understanding that fractions represent parts of a whole, students can connect the concept of fractions to other mathematical operations and develop a more comprehensive understanding of mathematics.

Overall, understanding that the whole matters when working with fractional parts is a crucial big idea underlying the concept of fractions. Teachers can promote this understanding by using real-world contexts, problem-solving tasks, and visual models to help students connect fractions to the whole. These activities can form the foundation for building conceptual understanding around fractions as part of a larger whole.

2.3.3. To Maintain Equivalence, the Ratio of the Related Numbers Must Be Kept Constant

A critical concept underlying fractions is that to maintain equivalence, the ratio of the related numbers must be kept constant. For example, equivalent fractions are fractions that represent the same portion of a whole but are written differently. To determine equivalent fractions, one must divide or multiply both the numerator and the denominator by the same number to maintain equivalence. This essential idea underpins many mathematical concepts beyond fractions, such as proportion, solving equations, and comparing and ordering numbers, and is necessary for success in higher-level mathematics.

Understanding the big idea of maintaining equivalence in fractions has been shown to be essential to comprehend basic algebraic concepts in higher-level mathematics (Dowker, 2004). When students do not master this concept at the early stages, they can face difficulties in their math education and struggle with more advanced levels. Therefore, it is critical to teach fraction concepts explicitly and facilitate meaningful connection with ratio and proportion concepts.

To support the teaching of maintaining equivalence in fractions, visual models such as area models, number lines, and fraction bars can help illustrate the concept for students and improve their understanding (Van de Walle et al., 2015). Recent research has validated the importance of using visual models to provide students with concrete representations to engage in meaningful discourse and connect math concepts to context-relevant problem-solving (Fuson & Tsao, 2016).

In addition, students are commonly instructed on how to multiply or divide by one in order to discover equivalent fractions ($\frac{2}{2}$, $\frac{3}{3}$, etc.). Understanding that $\frac{6}{8}$ is not equal to $\frac{3}{4}$ multiplied by two, for instance, requires students to comprehend the implicit 2-to-1 ratio. In Figure 2.5, a rectangle is divided into eighths, of which six are shaded. To establish equivalence, each pair of sections must be merged into a single unit. Then, instead of eights, there are fourths (separated by arrows) and only three shaded parts, as contrasted to six.

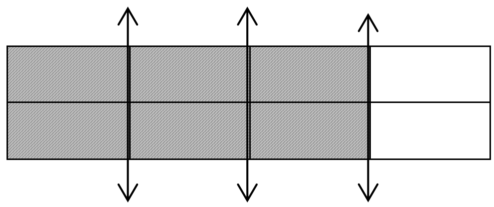


Figure 2.5. Equivalent fractions, $\frac{6}{8} = \frac{3}{4}$

2.3.4. The Properties (Distributive, Associative, and Commutative) that Hold for Natural Numbers, also Apply for Rational Numbers

Mathematical fundamentals include the properties of arithmetic operations such as addition, multiplication, subtraction, and division. They serve as a foundation for algebraic manipulation and mathematical reasoning. The distributive, associative, and commutative properties are crucial properties. These properties apply to natural numbers and assert that addition, multiplication, and division operate predictably. The commutative property of addition, for instance, states that altering the order of the numbers does not affect the sum. These properties are fundamental to mathematical reasoning and problem-solving.

These properties also hold for fractions, decimals, and other rational numbers, which is excellent news for students. Students must comprehend how these properties apply to rational numbers in order to manipulate meaningful numerical expressions and equations. The distributive property, for instance, pertains to both natural and rational numbers and states that multiplying a sum by a number is equivalent to multiplying each addend by that number separately and then adding the results. This concept is essential for refining fractional expressions. Similarly, the associative property states that the grouping of numbers in addition or multiplication has no effect on the result, while the commutative property states that the order of the numbers has no effect on the result. These concepts apply equally to rational and natural numbers (NCTM, 2021).

Prior study has prepared students to apply distributive, associative, and commutative properties to whole-number operations. In this study, students will understand that these properties apply to rational numbers and fractions. For example, $\frac{1}{3}$ of $\frac{3}{4}$ is equivalent to $\frac{3}{4}$ of $\frac{1}{3}$ (for a case of the commutative property). To calculate $\frac{5}{8}$ of 26, use a partial product such as $\frac{1}{2}$ (or $\frac{4}{8}$) of 26 plus $\frac{1}{8}$ of 26 (for a distributive property case).

2.4. Strategies Underlying Multiplication with Fractions

According to the preceding explanation and examples, big ideas are crucial for comprehension, and mathematizing the context is the core. This mathematizing process involves the development of strategies that students employ to solve multiplication problems involving fractions (Fosnot and Dolk, 2002). In this subject, students are expected to use various strategies, including:

2.4.1. Skip-Counting and/or Using Repeated Addition to Find a Fraction of a Whole

When students are introduced to multiplication with fractions, it is important to focus on strategies that will help them to develop a conceptual understanding of the process. Two strategies that have been shown to be effective are skip counting and using repeated addition. According to the National Council of Teachers of Mathematics (NCTM), these strategies can help students to see fractions as parts of wholes and can help them to make sense of multiplication with fractions (NCTM, 2020).

Skip counting, also referred to as repeated addition, involves adding a number successively to itself, which produces a product that is equal to the multiplication of that number and another natural number. For example, the multiplication problem 3×4 can be solved by skip counting 3 (one set of 3) four times to get 12. When working with fractions, students can use skip counting by recognizing that fractions represent parts of a whole. For example, to find $\frac{2}{3}$ of 12, students can skip count by adding $\frac{1}{3}$ of 12 two times: $\left(\frac{1}{3} \times 12\right) + \left(\frac{1}{3} \times 12\right)$. This strategy shows students that multiplication with fractions is a process of adding the fraction multiple times (NCTM, 2020).

Using repeated addition to find a fraction of a whole involves breaking down the fraction into smaller and more manageable parts. For example, to solve $\frac{3}{4} \times 16$, students can think of $\frac{3}{4}$ as $\frac{3}{4}$ of 4 and $\frac{3}{4}$ of 4 of another 4. By breaking the fraction down into smaller parts, students can use repeated addition to find the overall product. They can then add the two products together to get the final answer. This strategy can help students to understand multiplication with fractions conceptually and avoid relying solely on memorizing procedures (NCTM, 2020).

In conclusion, skip counting and using repeated addition are two strategies that can help students to develop a deeper understanding of multiplication with fractions. These strategies have been recommended by experts in mathematics education as effective and meaningful ways for students to approach fraction multiplication (NCTM, 2020). By using these strategies, students can build a strong foundation in math that can lead to improved performance and increased confidence in mathematical problem-solving. In this study, to calculate $\frac{7}{12}$ of 26 km running course, they will divide 26 into 12 equal parts to obtain $\frac{1}{12}$. Then, they will add: $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$.

2.4.2. Using Multiplication and Division to Make Equivalent Fractions

Using multiplication and division to make equivalent fractions is an important skill in elementary mathematics education. This skill involves using multiplication and division to multiply or divide both the numerator and denominator of a fraction by the same number, resulting in a fraction that is equal in value to the original but with different numerals. This process helps students to understand the concept of equivalent fractions and provides a foundation for subsequent skills such as adding and subtracting fractions. According to the Common Core State Standards Initiative (CCSSI), students should be able to use multiplication and division to make equivalent fractions by the end of fourth grade (CCSS, 2010).

To make equivalent fractions using multiplication, students need to find a common factor of the numerator and denominator and then multiply both by that factor. For example, to find an equivalent fraction of $\frac{3}{4}$, we could multiply both the numerator and denominator by 2 to get $\frac{6}{8}$, which is equivalent to $\frac{3}{4}$. Alternatively, we could multiply by 3 to get $\frac{9}{12}$, which is also equivalent to $\frac{3}{4}$. To make equivalent fractions using division, students need to find a common factor of the numerator and the denominator and then divide both by that factor. For example, to find an equivalent fraction of $\frac{6}{8}$, we could divide both the numerator and denominator by 2 to get $\frac{3}{4}$, which is equivalent to $\frac{6}{8}$ (CCSS, 2010).

It is important to help students understand the concept of equivalent fractions and not just memorize formulas. Making connections between multiplication and division and the concept of equivalent fractions can help students build a stronger foundation in math. Teaching students strategies such as finding common factors and using visualization tools like fraction bars can also help make the concept of equivalent fractions less abstract and more concrete. Using real-life examples, such as dividing a pizza into equal slices, can also help students connect with the concept of equivalent fractions (Wheaton et al., 2015).

In this study, when students realize that $\frac{7}{12}$ of a number is the same as $\frac{1}{12}$ of the number seven times, they begin to divide the number by its denominator and multiply it by its numerator. However, this strategy is not always effective. Before implementing this method, students should examine the numbers.

2.4.3. Using Landmark Fractions to Make Partial Products

Using landmark fractions to make partial products is an effective strategy for students to develop multiplication skills with fractions. This skill involves breaking down complex fractions into easier, more manageable fractions that can be multiplied mentally. A landmark fraction is a fraction that is convenient to use as it is easily recognizable and can simplify the process of multiplication. In elementary mathematics education, using landmark fractions to make partial products helps students to understand the concept of fractions as well as multiplication. According to the CCSSI, students in fifth grade should be able to apply the concept of using landmark fractions to make partial products to solve multiplication problems (CCSS, 2010).

To use landmark fractions to make partial products, students need to identify the landmark fraction that will simplify the multiplication. For example, to find the product of $\frac{2}{3}$ and $\frac{3}{4}$, students can recognize that $\frac{1}{2}$ is a landmark fraction as it is halfway between 0 and 1. This method helps students to understand that multiplication is commutative and that fractions can be simplified using techniques such as finding common denominators and using equivalent fractions (CCSS, 2010).

It is important to differentiate instruction based on students' readiness level when teaching the concept of using landmark fractions to make partial products. Some students may require more concrete tools such as fraction bars or models to understand the concept, while others may be ready to use abstract strategies such as mental math. Teachers can also encourage students to use estimation skills to check their answers and ensure they are reasonable. Finally, providing students with opportunities to apply the concept of using landmark fractions to make partial products to real-world scenarios can help students appreciate the relevance of the concept (NCTM, 2000).

In this study, calculating $\frac{7}{12}$ of 26 becomes simple when students consider landmark fractions. They could begin by calculating half of 26 (13), then calculate $\frac{1}{12}$ of 26 and combine the two partial products. This strategy relies on the distributive property of multiplication over addition.

2.4.4. Using Ratio Tables as a Tool for Making Equivalent Fractions

A ratio table is a helpful tool for keeping track of multiplicative pattern comparisons, according to Streefland (1991). In addition, according to Van Galen et al. (2008), the advantage of a ratio table is that each number has its own position and the unit of measurement must remain constant.

Using ratio tables as a tool for making equivalent fractions is a helpful strategy for students to develop a deep understanding of fractions. Equivalent fractions represent the same quantity but may be expressed in different ways. Students in elementary mathematics education must have a thorough understanding of equivalent fractions to develop and apply critical math skills. According to the CCSSI, students should be able to use equivalent fractions to solve problems involving multiplication and division of fractions by the end of the fifth grade (CCSSI, 2010). One strategy for finding equivalent fractions is using ratio tables.

A ratio table is a tool that displays equivalent ratios or fractions, where the numerator and denominator are multiplied or divided by the same factor to find equivalent pairs (Lazić et al., 2017). To use a ratio table to find equivalent fractions, the first step is to set up a table with the original fraction in the top row. Then, the student can multiply or divide the numerator and denominator by the same factor in each row to find an equivalent fraction. This technique can be repeated until the student finds the desired or simplest form of an equivalent fraction (Van de Walle et al., 2013).

Using ratio tables as a tool for making equivalent fractions helps students to understand the relationship between the numerator and denominator, as well as the concept of reducing fractions to their simplest form. It also helps students to develop problem-solving skills by analyzing data and making connections between mathematical concepts. As with any math technique, practice is essential for students to develop confidence and mastery.

Teachers can differentiate instruction by providing students with opportunities to use ratio tables and other fraction models as concrete tools to understand equivalent fractions. This approach helps students to develop a deep understanding of fractions before moving on to abstract strategies such as mental math. As students become proficient in using ratio tables, teachers can provide them with word problems or real-world scenarios to apply this concept and consolidate their learning (NCTM, 2014).

In conclusion, using ratio tables as a tool for making equivalent fractions is a valuable strategy in elementary mathematics education. This technique helps students to understand fractions, promote problem-solving skills, and make real-world connections. By providing students with opportunities to use ratio tables, teachers can facilitate the development of deeper mathematical reasoning skills. In this study, to find $\frac{1}{12}$ of 26, for instance, students could discover $\frac{1}{12}$ of 26 and halve it ($\frac{1}{4}$ of 26 = $\frac{1}{2}$ of 13), then find $\frac{1}{3}$ of $6\frac{1}{2}$ ($\frac{1}{12}$ of 26 = $\frac{1}{3}$ of $6\frac{1}{2}$ = $2\frac{1}{6}$). The foundation of this strategy is proportional reasoning.

2.4.5. Using the Standard Algorithm for Multiplication of Fractions: Ratio of the Product of Numerators to the Product of Denominators

The standard algorithm for multiplication of fractions involves multiplying the numerators of the fractions together and the denominators of the fractions together. This algorithm is a quick and efficient way to multiply fractions, but it is essential that students understand the mathematical reasoning behind it. The ratio of the product of the numerators to the product of the denominators represents a valid mathematical explanation of the standard algorithm for multiplication of fractions.

Understanding the mathematical reasoning behind the standard algorithm for multiplication of fractions can help students to develop a deeper understanding of mathematical concepts and improve their problem-solving skills (Son & Senk, 2010).

By understanding the ratio of the product of numerators to the product of denominators, students can apply this concept to various mathematical problems, including fractions with mixed numbers or improper fractions. For example, if a student has to multiply two fractions, $\frac{2}{3}$ and $\frac{3}{5}$, they can use the standard algorithm to obtain a product of $\frac{6}{15}$ or simplify it to $\frac{2}{5}$ by using the ratio of the product of the numerators to the product of the denominators (Leiva & Sarama, 2005).

It is essential to note that the standard algorithm for multiplication of fractions is not the only method for multiplying fractions, and students need to learn multiple strategies to solve fraction multiplication problems. However, the standard algorithm remains a vital tool for students in middle and high school to simplify and compare complex fractions (Van de Walle et al., 2013).

In conclusion, the standard algorithm for multiplication of fractions is a valuable mathematical tool that requires an understanding of the ratio of the product of numerators to the product of denominators. This technique can equip students with the skill of solving complex fraction multiplication problems using a quick and efficient method. However, educators should teach multiple strategies for solving fraction multiplication problems to ensure students have a deep understanding of fractions and the mathematical concepts involved.

In this study, students can learn the standard algorithm for fraction multiplication when the array model is used (Fosnot & Dolk, 2002; Tarlow-Hellman & Fosnot, 2007). Figure 2.6 depicts $\frac{2}{3} \times \frac{3}{4}$ as an example. The product is $\frac{6}{12}$. The denominators (3×4) formed the outer rectangle, and the numerators (2×3) made the inner rectangle.

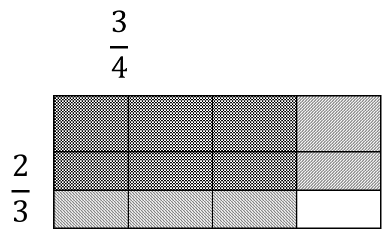


Figure 2.6. An array model for $\frac{2}{3} \times \frac{3}{4}$

2.4.6. Interchanging Numerators (or Denominators) to Simplify First When Multiplying

When multiplying fractions, the strategy of interchanging numerators or denominators can be a helpful tool for simplifying fractions before multiplication. This approach involves swapping the numerators and denominators of one or more of the fractions before performing the multiplication. For instance, when multiplying $\frac{2}{3}$ and $\frac{6}{8}$, students can interchange the numerators of one fraction with the denominator of the other, resulting in a calculation of $\frac{2}{8} \times \frac{6}{3} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2}$.

Interchanging numerators and denominators before multiplication is a useful strategy when dealing with complex fractions, such as mixed numbers, or when the fractions have different denominators. It is also a method of expressing the relationship between two fractions in a different form.

Teacher can introduce the interchanging numerators and denominators strategy during lessons on fraction multiplication as an alternative approach to solving fraction multiplication problems. Through guided practice, students can learn to simplify fractions using this method before proceeding to the multiplication stage. By doing so, students can gain a deeper understanding of fraction multiplication and improve their overall problem-solving skills.

In conclusion, the strategy of interchanging numerators or denominators before multiplication can be a useful tool for simplifying fractions and for expressing the relationship between two fractions in a different form. When taught as an alternative approach to solving fraction multiplication problems, this strategy can help students develop a deeper understanding of fraction multiplication and improve their problem-solving skills. Once students comprehend the basic procedure, in which the solution is the ratio of the product of the numerators to the product of the denominators, they can be encouraged to develop a strategy for exchanging the numerators. For instance, if the numerators in the previous problem are swapped, the solution will be simpler to calculate, i.e., $\frac{2}{3} \times \frac{3}{4} = \frac{3}{3} \times \frac{2}{4}$.

According to the big ideas and strategies underpinning multiplication of fractions, as described above, we designed learning exercises that allowed students to formalize their mathematical reasoning. Fosnot and Dolk (2002) argued that:

> "for children to become able to mathematize their world, they need to be allowed to do so in their own meaning-making ways as they are learning. They need to struggle with the big ideas and progressively refine their strategies to solve the problems." (p. 70)

The learning activities were designed so that students could analyze the relationships and develop their own procedures and strategies for mathematizing mathematics. Rather than expecting students to use a predefined algorithm to solve multiplication problems involving fractions, the activities should foster their progressive development. In addition, students must be guided to reconstruct their own mathematical understanding through contextual problems corresponding to its mathematical concepts (Streefland, 1991; Behr et al., 1994; Olive, 1999).

2.5. Types of Models for Fractions

Numerous types of models can assist students in learning and comprehending fraction multiplication. It is prudent to begin with general fraction models before moving on to fraction multiplication models. In addition to allowing students to choose their own models, several examples of models are described in detail and presented in a context that promotes comprehension. According to Van de Walle et al. (2008), models are essential to acquiring and comprehending fractions and fraction operations. Models can assist in the elaboration of concepts that are difficult to comprehend when provided only in symbolic form. In addition, models enable students to examine problems from a number of angles and perspectives, and certain models adapt themselves to specific situations more easily than others.

When applied properly, models can help students clarify ideas that are usually misinterpreted in a purely symbolic form. When completing the same activity with two alternative representations and asking students to establish links between them, it may be beneficial to ask students to do so. Different representations offer unique learning opportunities.

The area/array model, for instance, supports students in conceptualizing the parts of a whole. This model involves dividing a shape or region into equal parts and shading or coloring the appropriate number of parts to represent a given fraction. This model can be used to compare and add fractions, as well as to understand equivalent fractions.

Another type of model is the number line/length model, which represents fractions along a line with a starting point and an endpoint. The number line model can be used for comparing fractions, identifying equivalent fractions, and adding and subtracting fractions with like denominators (Bakker & Gravemeijer, 2004; Gunderson, et al., 2019). This model explains that there is always another fraction between any two integers, an essential concept that is sometimes overlooked in fractions instruction. Significant real-world contexts must also be used to teach students fractions (Cramer & Whitney, 2010). Suppose students are asked, for instance, who runs the farthest. In that case, a number line is more likely than an area model to facilitate their understanding.

The set or collection model is also a useful tool for understanding fractions. This model involves representing a fraction as a subset of a larger set or collection. For example, a fraction such as $\frac{3}{8}$ can be represented as three out of eight objects in a collection. The set model can be used to compare, add, and subtract fractions with like denominators.

Additionally, the ratio model is another option for representing fractions. The ratio model involves comparing two quantities, such as the ratio of boys to girls in a classroom. This type of model can be used to introduce the concept of fractions and help students understand the relationship between fractions and the number system (Graeber et al., 2018).

Furthermore, there are models that represent fractions in more specialized contexts such as money, measurement, and geometry. The money model involves representing fractions as money amounts, which can be particularly helpful when students begin working with mixed numbers. The measurement model involves using fractions to measure quantities, such as time or distance, and can be used to introduce the concept of mixed numbers (Lamon, 2012). Lastly, the geometry model represents fractions as shapes, such as circles or rectangles, and can be used to introduce the concept of improper fractions (Van den Heuvel-Panhuizen, 2003).

Overall, the use of different types of models can greatly enhance understanding and learning of fractions. Teachers can use these models to help students who struggle with abstract concepts and to provide visual and concrete representation of complex mathematical concepts. This section discusses three types of models: area/array, length/number line, and set. In addition, ratio table is also added in this subchapter as it was also used in this study as one of the fraction models. Table 2.1 summarizes each model type, including definitions of their wholes and related parts according to Van de Walle et al (2013).

Table 2.1. Models for fractions concepts and how they compare (Van de Walle et al., 2013)

Model	What Defines the Whole	What Defines the Parts	What the Fraction Means
Area/Array	The area of the defined region	Equal area	The part of the area covered, as it relates to the whole unit

Length/Number Line	The unit of distance or length	Equal distance/length	The location of a point in relation to 0 and other values on the number line
Set	Whatever value is determined as one set	Equal number of objects	The count of objects in the subset, as it relates to the defined whole

2.5.1. Area/Array Model

One approach to teaching fractions that has gained popularity in recent years is the area/array model. This model represents fractions using rectangular arrays of squares or circles, where the fractional part of the whole is shaded in a specific color. For example, a half could be represented by shading in half of a square or circle, while a third would be represented by shading in one of three equal parts (Cramer, Post, & Currier, 2016).

The area/array model can be used to introduce a variety of fraction concepts and operations. Students can use this model to explore the relationship between fractions and decimals, as well as understand equivalent fractions and the concept of adding and subtracting fractions with like or unlike denominators (Barrett, 2015). This model can also be used to introduce multiplication and division of fractions, as students can see how multiplying the denominator and numerator by the same factor can represent a larger or smaller fraction of the whole (Clements & Sarama, 2009).

One of the advantages of the area/array model is that it provides a concrete and visual representation of fractions, which can make abstract concepts more accessible to students. Students can also use this model to develop a deep understanding of fraction concepts, as they can see how different fractions relate to each other and to the whole. In addition, the use of the area/array model can help students develop spatial reasoning skills, as they are required to visualize and manipulate the fractions and arrays (Barrett, 2015).

According to Van de Walle et al. (2013), when children engage in a sharing tasks, the knowledge that fractions represent parts of an area is a fundamental concept (2008). There are numerous ways to represent these area models. Circular fraction piece models are generally regular and have the benefit of underlining both the part-whole concept of fractions and the meaning of a part in relation to the whole. Similar area models can be created using rectangular regions, geoboards, drawings on grids or dot paper, pattern blocks, and folding paper (Figure 2.7). In this figure, Van de Walle et al. (2013) illustrate the many types of area models (p. 293).

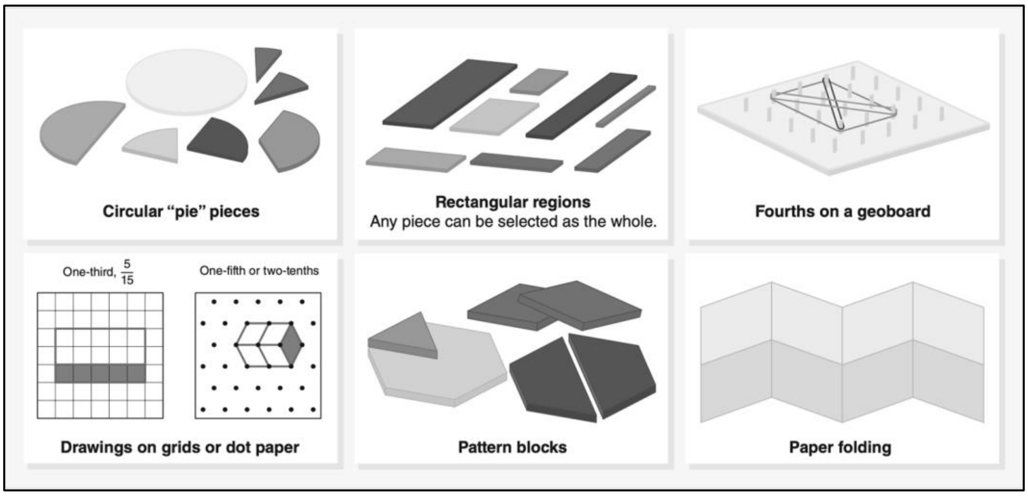


Figure 2.7. Area or region models for fractions (Van de Walle et al., 2013, p. 293)

In summary, the area/array model is a useful tool for teaching fractions and its operations. This model provides a concrete and visual representation of fractions, which can help students develop a deep understanding of the concepts. The use of the area/array model can also help develop spatial reasoning skills, making it a valuable addition to any fraction instruction.

2.5.2. Length/Number Line Model

Another approach to teaching fractions is the length/number line model. This model represents fractions on a number line, where the whole is divided into equal intervals, and fractions are represented by counting the number of intervals between two natural numbers. For example, $\frac{1}{2}$ would be represented by counting the two equal intervals between 0 and 1, while $\frac{2}{3}$ would be represented by counting the three equal intervals between 0 and 2 (Sowder, 2001).

The length/number line model can be used to introduce a variety of fraction concepts and operations. Students can use this model to understand equivalent fractions, as they can see how different fractions can represent the same point on the number line. This model can also be used to introduce adding and subtracting fractions with like or unlike denominators, as students can use the number line to find a common denominator and add or subtract the numerators (Lamon, 2012).

One advantage of the length/number line model is that it allows teachers to integrate fractions with other mathematics concepts, such as measurement and money (Sowder, 2001). Students can use the number line to understand the relationship between fractions and decimals, as well as to make connections between fractions and proportional reasoning. In addition, the use of the length/number line model can help students develop a more precise understanding of fractions, as they are required to locate fractions on an exact point on the number line (Lamon, 2012).

Length models, as opposed to area models, compare lengths or measures. As described in Figure 2.8, either lines are drawn and subdivided, or the lengths of physical objects are compared. Even though length models are crucial for students to comprehend fractions, they are rarely utilized in classrooms. Recent evaluations of fractions research (Petit et al., 2010; Siegler et al., 2010) indicate that the number line assists children in understanding fractions as numbers and developing additional fraction concepts.

One of the most prevalent applications of fractions in the actual world is closely linked to linear model. Not only does the number line emphasize that a fraction is a single number, but it also allows comparisons of its relative size to other numbers, which is not possible with area models. Additionally, the number line highlights that there is always one more fraction between any two fractions.

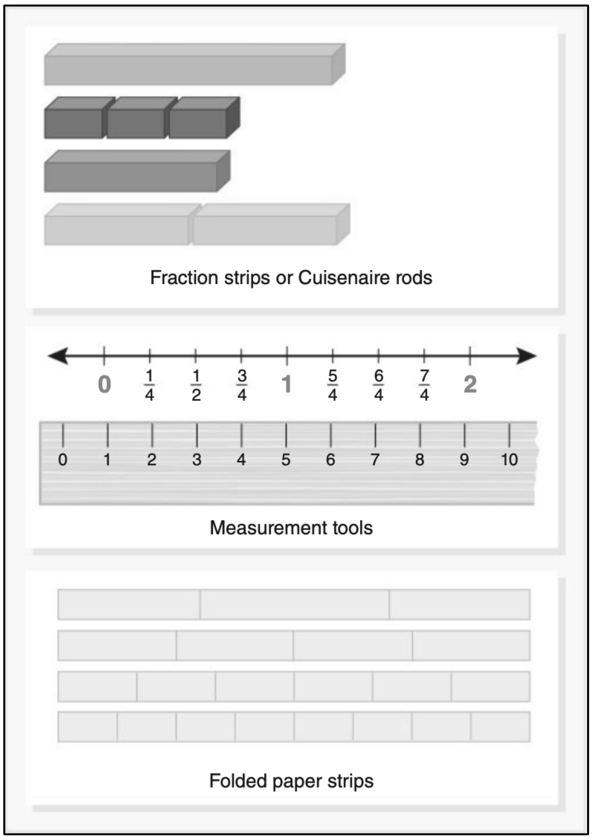


Figure 2.8. Length or measurement models for fractions (Van de Walle et al., 2013, p. 294)

In summary, the length/number line model is a useful tool for teaching fractions and its operations. This model provides a concrete and visual representation of fractions on a number line, which can help students develop a more precise understanding of the concepts. The use of the length/number line model can also help students integrate fractions with other mathematics concepts and develop proportional reasoning skills.

2.5.3. Set Model

Another approach to teaching fractions is the set model. This model represents fractions as parts of a set, where the whole is represented by a set of objects and the fraction is represented by a subset of those objects. For example, if the whole set contains 10 objects and the fraction is $\frac{3}{10}$, then the subset would contain 3 of the 10 objects (Behr et al., 1992).

The set model can be used to introduce a variety of fraction concepts and operations. Students can use this model to understand equivalent fractions, as they can see how different fractions can represent the same subset of objects. This model can also be used to introduce adding and subtracting fractions with like or unlike denominators, as students can use the set model to find a common denominator and add or subtract the subsets (Lamon, 2012).

One advantage of the set model is that it can be used to help students develop a conceptual understanding of fractions. By seeing fractions as parts of a whole, students can better understand the meaning of fractions, as well as relate them to real-world situations (Behr et al., 1992). In addition, the use of the set model can help students develop a visual image of fractions, which can help them in their problem-solving abilities (Lamon, 2012).

The set model also provides opportunities for students to use manipulatives, such as fraction bars or circles, to represent fractions. These manipulatives can help students visualize fractions and operations with fractions (Behr et al., 1992). In addition, the use of manipulatives can help students develop a deeper understanding of the concepts, as they are able to physically manipulate and see the relationships between fractions (Lamon, 2012).

Fractional parts are subsets of the whole, which is regarded as a collection of objects in set models. For instance, five items comprise one-third of a collection of fifteen. In this instance, the group of 15 represents the whole or 1. Some students find set models problematic since they refer to a collection of counters as a single thing. Typically, students focus on the size of the set rather than the total number of identical sets. A set of four counters is one-third of the whole, not one-fourth, because the whole is composed of three equal sets. On the other hand, the set model facilitates the formation of substantial linkages between fractions and ratio concepts and their numerous practical applications. Figure 2.9 depicts a number of set models for fractions.

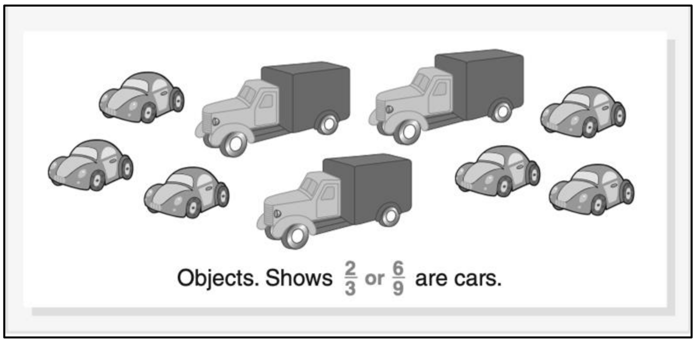


Figure 2.9. Set models for fractions (Van de Walle et al., 2013, p. 295)

In summary, the set model is a useful tool for teaching fractions and its operations. This model provides a concrete and visual representation of fractions as parts of a whole, which can help students develop a conceptual understanding of the concepts. The use of the set model also provides opportunities for students to use manipulatives to represent fractions and operations with fractions.

2.5.4. Ratio Table Model

The ratio table model is another approach to teaching fractions, which represents fractions as ratios of two numbers (typically natural numbers) in a table format. This model can be used to introduce a variety of fraction concepts and operations, including equivalent fractions, adding and subtracting fractions with like and unlike denominators, and multiplying and dividing fractions (Steen, 1991).

The ratio table model is a non-visual model, which means that it does not rely on a visual representation of fractions. Instead, this model uses a table format to show the relationship between two numbers and how they are related in a fraction. For example, if the whole is divided into 8 equal parts and 3 of those parts are shaded, the fraction can be written as $\frac{3}{8}$, with the 3 representing the number of shaded parts and the 8 representing the total number of equal parts (Clements, 2003).

One advantage of the ratio table model is that it provides a clear and organized way to show the relationship between two numbers in a fraction. Students can use this model to develop an understanding of the different parts of a fraction, as well as the relationship between the numerator and denominator (Clements, 2003). In addition, the ratio table model can be used to introduce complex fraction concepts, such as mixed numbers and improper fractions (Steen, 1991).

Another advantage of the ratio table model is that it can be used to help students develop problem-solving skills related to fractions. By using the table format to break down complex fractions and problems, students can develop a systematic approach to solving fraction problems (Clements, 2003). In addition, the ratio table model can be used to help students build fluency with fractions, as they practice converting between different forms of fractions and using the table to perform operations (Steen, 1991).

In summary, the ratio table model is a useful tool for teaching fractions and its operations. This model provides a non-visual representation of fractions as ratios of two numbers, which can help students develop a conceptual understanding of the concepts. The use of the ratio table model also provides opportunities for students to develop problem-solving skills related to fractions.

2.6. Realistic Mathematics Education (RME) and Mathematizing Processes

The Wiskobas project in the Netherlands established the instructional approach known as Realistic Mathematics Education (RME) as a counter to the New Math or Mathematics Modern curriculum that was prevalent at the time (see Freudenthal, 1973, 1991; Treffers, 1987, 1991; Streefland, 1991; Gravemeijer, 1994, 1997; van den Heuvel-Panhuizen, 1996; Klein, 1998). The term realistic comes from a categorization by Treffers (1987), which distinguishes four ways to teach mathematics: mechanistic, structuralistic, empiristic, and realistic. Later on, a realistic approach to mathematics education evolved to be known as Realistic Mathematics Education. This method was founded on Freudenthal's understanding of mathematics as a human activity, which was published in Freudenthal's 1973 book.

The fundamental tenet of RME is that students, while being supervised by an adult (teacher), should be provided with the chance to reinvent mathematical concepts. Furthermore, students' informal mathematical knowledge may be used as a foundation upon which to build their formal mathematical understanding (Treffers, 1991a). It means that students may utilize their informal knowledge to reinvent mathematics by participating in various activities that include finding solutions to real-world problems that arise in the classroom setting. According to this point of view, learning mathematics would consist of a great deal of interaction, with teachers being required to build upon the concepts presented by students.

A realistic approach considers mathematics to be an activity rather than a subject. Doing mathematics is a fundamental component of learning mathematics, and one of the most important aspects of doing mathematics is finding solutions to problems that arise in real-world contexts (Gravemeijer, 1994). Freudenthal (1973) describes the activity that we engage in as an activity of solving issues, of seeking for problems, and also an activity of organizing a subject matter. This may be an issue that arises from the actual world, which, in order to be solved, has to be arranged in accordance with mathematical patterns. There is also the possibility that it is a mathematical issue, in which case new or old, either your own or those of others, need to be structured according to new concepts in order to be better understood in a larger context.

RME is an approach to teaching math that emphasizes the use of real-world contexts to develop deep and meaningful understanding of mathematical concepts and procedures (Freudenthal, 1983). One of the central ideas of RME is the use of mathematizing processes to help students make sense of real-world problems and situations, and to develop a flexible and transferable understanding of mathematical ideas. The term mathematizing refers to this type of organizational action (Treffers, 1991a; Gravemeijer, 1994; 1997). Mathematizing is a crucial step in the mathematical education process, according to Freudenthal, and this is for two reasons. To begin, the process of mathematizing is not only the primary focus of the work of mathematicians, but it also acquaints students with a mathematical perspective on the everyday problems they face. For instance, the mathematical activity of addressing contextual issues requires a mathematical mindset. This mindset involves being aware of the possibilities and limitations of a mathematical approach, as well as being able to recognize when such an approach is suitable and when it is not.

Mathematizing processes involve several steps, including:

- Identifying a real-world problem or situation that requires mathematical thinking and analysis.
- Representing the problem mathematically, using symbols, equations, diagrams, graphs, or other tools.
- Solving the mathematical problem, using relevant mathematical concepts, procedures, and strategies.
- Interpreting the solution in the context of the original problem, and validating the solution to ensure that it makes sense.

Through mathematizing processes, students are able to develop a coherent and integrated understanding of mathematical concepts and procedures, rather than simply memorizing calculations or procedures (Van den Heuvel-Panhuizen, 2003). This approach to math education supports the development of mathematical literacy, which is characterized by the ability to use and apply mathematical knowledge in a variety of real-world settings (NCTM, 2000).

RME and mathematizing processes have been found to be effective in promoting student learning and engagement in math. In a study of RME in the Netherlands, students who received RME instruction scored significantly higher on a national math test than students who received traditional math instruction (Van den Heuvel-Panhuizen & Drijvers, 2014). Another study found that RME helped students to see the relevance and applicability of math to their everyday lives, leading to higher levels of motivation and engagement (Sowder et al., 2013).

In mathematics, the second-to-last step is called formalizing, and it is accomplished by axiomatizing. This terminal point should not serve as the beginning point for the mathematical teaching we provide, as this is how mathematics is often presented in more traditional settings. Freudenthal contends that beginning with axioms is an anti-pedagogical inversion because the procedure by which mathematicians reach their conclusions is the opposite of what is taught in schools. In this context, he proposes that mathematics instruction should be structured as a process of guided reinvention. This means that students should be given the opportunity to experience a process that is analogous to the one that mathematicians used when they first conceived about mathematics.

The process of conceptual mathematization in RME is illustrated in Figure 2.10 by De Lange (2006). This image clarifies not only why real-world context serves as an essential foundation for studying mathematics but also why it is so crucial to begin with. According to De Lange (2006), the process of producing mathematical concepts and ideas originates with the real world, and at the conclusion of the process, we need to reflect the answer back to the real world. So, the process of teaching mathematics consists of taking examples from the real world, abstracting mathematical concepts from those examples, and then applying those abstract concepts back to the real world. The end result of all of these processes was the conceptualization of mathematics.

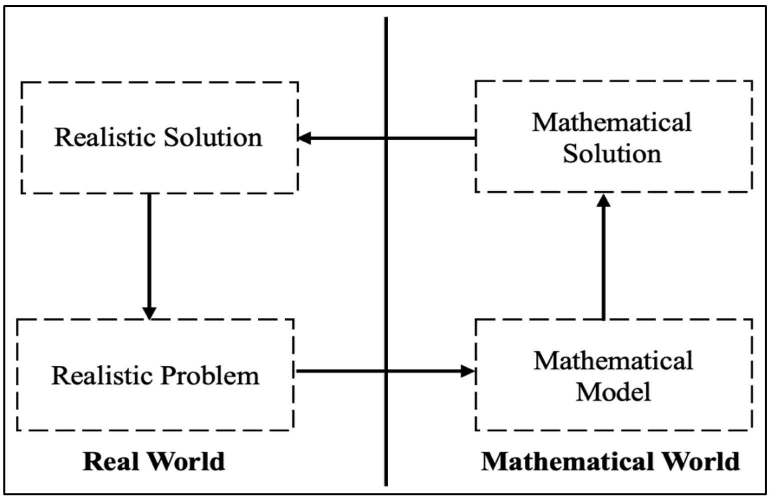


Figure 2.10. Conceptual mathematization (De Lange, 2006, p. 17)

Gravemeijer (1994, 1997) identifies three main heuristic concepts of RME that should be incorporated into the design of any instructional materials (see also Gravemeijer & Terwel, 2000). These include guided reinvention through progressive mathematization, didactic phenomenology, and self-developed or emergent models to facilitate creative problem-solving and innovation.

In conclusion, RME and mathematizing processes are promising approaches to math education that promote deep and meaningful understanding of mathematical concepts and procedures. By using real-world contexts, students are able to develop a flexible and transferable understanding of math that prepares them for success in a range of academic and real-world settings.

2.6.1. Guided Reinvention through Progressive Mathematization

In RME, guided reinvention through progressive mathematization is a central teaching strategy. This approach involves guiding students through the mathematizing processes in a series of increasingly complex and abstract situations, with the goal of building understanding and fluency with mathematical concepts and procedures (Gravemeijer & van Eerde, 2009). Teachers provide scaffolding and support as students work through the mathematizing processes, but also encourage students to make their own discoveries and connections.

Gravemeijer and van Eerde (2009) suggest that in order for guided reinvention through progressive mathematization to be effective, students must engage in four main types of activities: (1) solving problems, (2) reflection and sharing, (3) guided reinvention, and (4) application. In this approach, students learn math concepts by solving progressively more complex problems and reflecting on their solutions. The teacher then guides students to reinvent mathematical concepts and procedures on their own, using their reflections and past problem-solving experiences as a basis. Finally, students apply their learning to new problems and contexts.

Research suggests that this approach can be effective in improving students' mathematical understanding and fluency. For example, O'Connor and Michaels (2017) found that using guided reinvention through progressive mathematization in a high school geometry class improved students' understanding of mathematical concepts and promoted deeper learning. Similarly, Maher et al. (2010) found that using this approach in middle school mathematics led to improved problem-solving skills and conceptual understanding.

With RME, the real-world problem is initially studied informally to formalize it mathematically, as stated by De Lange (2006). This involves setting a strategy, categorizing the problem, and searching for mathematical characteristics and patterns. Eventually, new mathematical ideas will emerge from this exploratory phase, which depends mainly on intuition. First, RME's guiding philosophy for instructional design is guided reinvention through progressive mathematization, which is based on these requirements.

In accordance with the guided reinvention concept, students should be allowed to engage in an activity analogous to the development of mathematics (Gravemeijer 1994, 1999). According to this concept, a learning path must be planned, enabling students to discover the required mathematics independently. The developer or designer begins constructing the learning path by doing a mental experiment in which he or she speculates how he or she may have arrived at the result. According to Gravemeijer (1994), the focus of the conjectured learning trajectory should not be on creating new mathematical ideas or findings but rather on the nature of learning itself. This involves allowing students to learn on their own terms, at their own pace, and assume responsibility for the information they acquire. This means that during the instructional learning process, students should be provided with opportunities to construct their own mathematical knowledge.

Gravemeijer (1994, 1997) suggests using two tools to implement the reinvention concept. One benefit of studying mathematics' history is that it sheds light on the evolution of certain areas of expertise. With this information, the developer or instructional designer may be able to map out the intermediary stages necessary to reimagine the required mathematics. This allows students to benefit from the efforts of professional mathematicians. Second, the possibility for reinvention can be created by posing a contextual problem with a variety of informal solution techniques, and then proceeding to mathematize comparable answer procedures. Developers/instructional designers need to seek contextual issues with several viable solutions to achieve this, with a particular emphasis on problems that, when taken as a whole, provide a path to learning through progressively more mathematical content.

According to Gravemeijer (1999), the reinvention process is one of progressive change since it is part of a long-term learning process. Focusing on guided exploration is essential, and the intermediate steps must always be seen in the context of the whole picture rather than as the final. This perspective may be realized through the development and instructional design of an appropriate series of contextual problems. Comparing the realistic approach and information processing in terms of the reinvention process will help us grasp the guided reinvention concept. As described by Gravemejer (1994), the information processing method treats mathematics as a prefabricated system with broad applicability and considers mathematics education as the deconstruction of formal mathematical knowledge into the acquisition of processes and their subsequent application. In contrast, the realistic approach stressed that learning mathematics involves engaging in mathematical activity, which includes solving daily problems (see Figure 2.11).

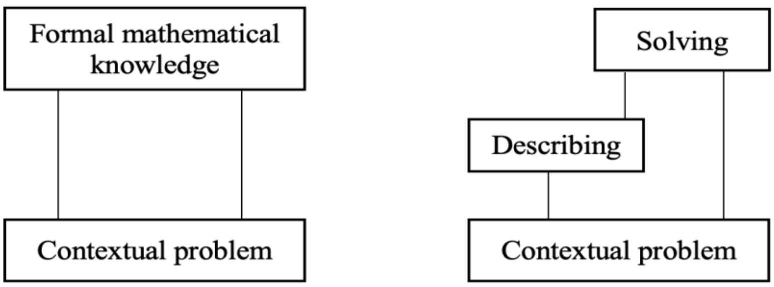


Figure 2.11. Processes of mathematical learning in information processing and realistic approaches (Gravemeijer, 1994, p. 93)

The first model describes the process of finding a solution to a contextual problem by utilizing the information gained from the formal mathematical study. In the first phase, the issue is transformed into a mathematical problem (using mathematical language), followed by the solution of the mathematical problem through the appropriate mathematical tools. At the conclusion, the mathematical answer is recast within the parameters of its initial application. Gravemeijer (1994) has some criticism for this approach because to solve the problem, there is a reduction in the amount of information. The reduction of information that results from the transformation of a contextual problem into a mathematical problem is because doing so will result in many components of the original problem being eliminated. An interpretation is required whenever the mathematical answer is re-implemented into its original setting. On the other hand, the variables that were eliminated entirely had to be taken into consideration once again. What ends up happening, however, is that the recommendation derived from the mathematical solution does not truly suit the original situation. This is something that happens rather regularly. In addition, the success of this approach in resolving the issues is attributable to the identification of problem categories and the establishment of regular processes.

In the second model, the process of finding a solution to a problem similarly goes through three phases. First, the contextual problem is described more formally. Then, the problem at this level is solved. Finally, the answer is translated back into the context. Yet, because the realistic approach to mathematics instruction is based on human activity, it means that the three activities have a totally different meaning than those in the first model.

Gravemeijer (1994) outlines the advantages of using a realistic approach to solve problems, including:

- The actual meaning lies not in the use of mathematical tools but rather in the context of the problem itself,

- Instead of following the standard procedures, problem-solving is carried out in an informal environment,
- The problem is explained, which allows students to find a solution,
- Students will have a better understanding of the problem if they schematize the problem situations and identify the main relationships in those situations, and
- Because the symbols are meaningful, translation and comprehension of the solution are facilitated.

Thus far, it appears that mathematizing is a critical activity in RME. According to Freudenthal (1991), Streefland (1991), Gravemeijer (1994), and Schoenfeld (1994), the five mathematizing processes are modeling, symbolizing, generalizing, formalizing, and abstracting.

1. *Modeling* involves creating a mathematical representation of a real-world problem or situation. This might involve creating a diagram, using objects to represent quantities, or developing a mathematical formula based on real-world data. By modeling, students are able to make connections between the mathematical concepts they are learning and the real-world contexts in which they might apply them. At this level, the model is termed a situation-specific context model (Gravemeijer, 1994). For instance, in this study's setting of exploring playgrounds, an array model was utilized to depict the image of a playground area. According to Gravemeijer, this picture can serve as a model for formal mathematical reasoning (p. 100). The modeling strategy is more comprehensive.

2. *Symbolizing* is the process of representing mathematical concepts using symbols, notation, and algebraic expressions. This allows students to work with abstract mathematical ideas and to communicate their thinking more efficiently. By symbolizing, students are able to move beyond concrete representations of math concepts and toward more general and abstract ideas. According to Keijzer (2003), symbolization is the process of utilizing labels to explain a situation while simultaneously generating a new context. For example, in $\frac{1}{8}$ of a running route, the

‘8’ indicates that the route should be divided into eight pieces, and the ‘1’ indicates that one of those eight sections is measured. Discovering appropriate representations of the context in which symbols are generated is the process of symbolizing (Lienchevski, 1995). Gravemeijer (1998) discovered that a model might be transformed into a symbol. The ‘8’ in $\frac{1}{8}$ of a running route, for example, denotes the entire running route split by eight. In addition, the divided running route serves as a model for fractions.

3. *Generalizing* involves identifying patterns and structures in mathematical situations and extending them to new situations. This allows students to develop a deep and flexible understanding of mathematical concepts, rather than simply memorizing procedures. By generalizing, students are able to recognize similarities and connections between seemingly different math concepts and apply them in novel situations. The process of generalizing mental constructions is initiated by pattern recognition (Linchevski, 1995). At this time, students will understand the rules applicable to various things. For instance, the definition of $\frac{1}{8}$ of a running route can be applied to indicate taking one from an eight-part whole. In other words, generalization is the process of generating a (mental) construction.

4. *Formalizing* is the process of creating formal mathematical definitions, rules, and procedures based on the structures and patterns students have identified through modeling, symbolizing, and generalizing. This allows students to develop a more rigorous and precise understanding of mathematical concepts and procedures. By formalizing, students are able to take their informal and intuitive understanding of math concepts and develop a more formal and rigorous understanding. According to Gravemeijer (1994), formalizing is an extension of generalizing, where formalizing refers to an all-encompassing procedure or standard. It can be applied to a range of mathematical problems (p. 409). Consequently, formalization is the act of articulating mathematical relationships through symbols.

5. *Abstracting* involves identifying the essential features of a mathematical situation and articulating them in the most general terms possible. This allows students to generalize mathematical ideas and apply them in even more abstract and general contexts. By abstracting, students become more proficient in identifying and applying mathematical structures and patterns in a wide range of situations. Abstracting is the conversion of a general technique into a property (Mason, 1989). Mason's concept of shift is similarly related to Van Hiele's (1986) notion of connecting mathematical concepts through an abstract level. In addition, students will link the higher and lower levels of thought at this stage (Streefland, 1991).

Modeling, symbolizing, generalizing, formalizing, and abstracting are key aspects of mathematizing in RME (Streefland, 1991). By engaging in these processes, students will be able to develop a deep and flexible understanding of mathematical concepts and procedures, and to apply them in a variety of real-world contexts. In addition, because the process of mathematization is so essential to the development of students' knowledge from their thinking (Freudenthal, 1968; Treffers, 1991a), it is essential to begin the process by mathematizing those contextual issues that arise from student's day-to-day experiences in order to get the process off to a successful start. Treffers (1991a) refers to this process as horizontal mathematization. When students engage in this behavior, they are given the chance to address problems arising from their setting by employing informal language. In the long run, once the students have been exposed to comparable processes (such as simplifying and formalizing), the informal language will grow into a more formal or standardized language. This change will take place when the students have undergone similar procedures. After going through all of these steps, the students will finally arrive at an algorithm. Vertical mathematization is the term for applying mathematics to the mathematical subject matter (Treffers, 1987, 1991a). When it comes to mathematization, Freudenthal (1991) proposes a distinction between horizontal and vertical mathematization:

"Horizontal mathematization provides a bridge between the real world and the symbolic one. Vertical mathematization is the process by which symbols are formed, reshaped, and manipulated in a mechanical, understanding, and reflective manner, separate from the reality in which one really lives and acts. The world of life is what is experienced in regard to reality (in the way I used the phrase earlier); similarly, the world of symbols is what is experienced in reference to abstraction" (p.93).

In his study from 1987, De Lange makes a more comprehensive differentiation between horizontal and vertical mathematization depending on the kind of operations involved. Horizontal mathematization entails a number of activities, including the following: recognizing specific mathematics in a more general context; schematizing; formulating and visualizing a problem in a variety of different ways; finding relations; finding recurring patterns; identifying similarity aspects in a variety of problems; transferring a real-world problem to a mathematical problem; and transferring a real-world problem to a mathematical model that is already known. Meanwhile, activities included in vertical mathematization include describing a relation in a formula, demonstrating regularities, refining and changing models, employing various models, merging and incorporating models, articulating new mathematical ideas, and generalizing.

Figure 2.12 describes the mathematization process, including horizontal and vertical mathematization. Students engage in horizontal mathematization when they express contextual problems in their own informal ways to find solutions to those problems. If the students are able to solve the problems by employing mathematical language or by discovering an algorithm, then this process demonstrates vertical mathematization.

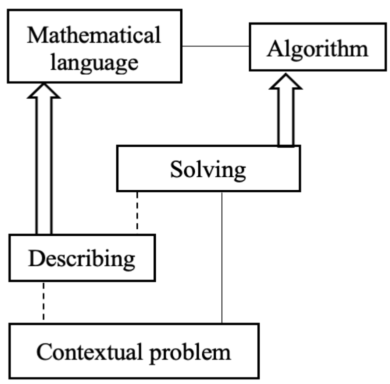


Figure 2.12. Horizontal Mathematization (------); Vertical Mathematization (⇨)
(Gravemeijer, 1994, p. 93)

If students are able to (re) build the formal mathematical knowledge as a result of this learning process, it indicates that the students have engaged in the process of reinvention. Gravemeijer (1994) provides a diagrammatic representation of this process in the following Figure 2.13. In spite of the fact that the process of reinvention as a single-directional arrow, in practice it is an iterative process. In other words, prior to re-inventing formal mathematical knowledge, students go through the processes of explaining and finding solutions to contextual problems with similar technique solutions. This is done before students move on to the next step of the process, which is to reinvent the formal mathematical knowledge. Throughout these procedures, students formalize the informal strategies they have developed into some type of mathematical language or algorithm.

Mathematizations may be constructed both horizontally and vertically when using the realistic approach, which is utilized to develop the long-term learning process. In this phase, the students will begin with contextual issues, as well as unique and informal knowledge and strategies. After that, they are required to develop formal mathematics by first mathematizing the contextual problems (in a horizontal manner) and then mathematizing the solution processes (vertically).

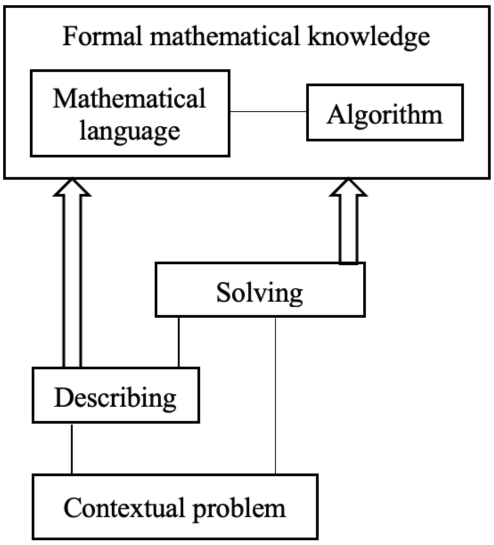


Figure 2.13. Reinvention process (Gravemeijer, 1994, p. 94)

In conclusion, guided reinvention through progressive mathematization is a central teaching strategy in RME, involving problem-solving, reflection and sharing, guided reinvention, and application. This approach can be effective in promoting deeper learning and improving students' understanding and fluency with mathematical concepts and procedures.

2.6.2. Didactical Phenomenology

Didactical phenomenology is a key theoretical framework used in Realistic Mathematics Education (RME) for understanding how students interact with mathematical concepts. This framework is based on the idea that students' learning experiences are shaped by their perceptions, emotions, and beliefs. According to didactical phenomenology, students' interactions with mathematical objects are influenced by their prior experiences and expectations, as well as by the cultural norms and values they bring to the classroom (Schmidt, 2011).

In RME, didactical phenomenology is used to design learning environments that support students' understanding of mathematical concepts. This approach involves using student-centered tasks that provide opportunities for students to engage with mathematical concepts in meaningful ways. For example, students may be presented with real-world problems or situations that require them to use multiple strategies to solve problems. The goal of this approach is to support students' construction of mathematical knowledge, rather than simply learning procedures or algorithms (Gravemeijer & Cobb, 2006).

Freudenthal (1973) supported didactical phenomenology as an alternative to the anti-didactic inversion. This suggests that to learn mathematics, we need to begin with situations that are significant for the student, want to be structured, and trigger the learning processes. In the field of didactical phenomenology, scenarios in which a particular mathematical topic is applied must be researched for two reasons (Gravemeijer, 1994, 1999). To begin, the goal is to demonstrate the types of applications that must be anticipated in the field of education. The second step is to evaluate whether or not these areas of impact are suitable for a procedure that involves the progressive mathematization of concepts.

According to Gravemeijer (1994, 1999), the objective of a phenomenological investigation is to identify problem situations for which situation-specific approaches can be generalized, as well as to identify problem situations that can arouse conceptual solution procedures that can be used as the basis for vertical mathematization. This objective may be understood by seeing the historical development of mathematics as an evolution from the solution of practical problems. Finding the contextual problems that contribute to this dynamic process is one way for us to fulfill this aim in the classroom when it comes to teaching mathematics.

The provision of students with contextual problems drawn from situations that are real to them and meaningful is a requirement of the didactical phenomenology principle, which carries with it the notion that the developer or instructional designer must do so. Nonetheless, there are occasions when mathematics instructors are confused about what the designation real or realistic in RME means. They understand it to refer to truly actual things or situations occurring in the surrounding environment.

Research suggests that didactical phenomenology can be an effective approach for promoting students' mathematical learning. For example, van de Walle et al. (2016) found that using a didactical phenomenology approach in a high school algebra course improved students' understanding of algebraic concepts and procedures. Similarly, Kieran (2007) found that using this approach in middle school geometry led to improved learning outcomes and helped students develop more positive attitudes towards mathematics.

In Didactical phenomenology, mathematical concepts are seen as having both an objective and a subjective dimension. The objective dimension refers to the mathematical structures and concepts that exist independently of human experience, while the subjective dimension refers to the ways in which students perceive, interpret, and make meaning of these structures and concepts. According to Didactical phenomenology, the subjective dimension is crucial for understanding how students interact with mathematical concepts and for designing effective learning environments (Boaler, 2016).

One of the key principles of didactical phenomenology is the idea of horizontal and vertical mathematization. Horizontal mathematization refers to the process of enriching students' experience of mathematics by connecting it to other areas of knowledge and experiences, such as science, art, and culture. Vertical mathematization refers to the process of connecting mathematical concepts across different levels of abstraction and complexity, building on students' prior knowledge and experiences (Gravemeijer, 2008).

Research has shown that using didactical phenomenology in mathematics instruction can have significant positive effects on students' learning outcomes. For example, a study by Nieveen et al. (2004) found that using this approach in primary school mathematics improved students' understanding of mathematical concepts and their ability to solve mathematical problems. Similarly, a study by Laborde (2010) found that using didactical phenomenology in high school geometry led to significant improvements in students' conceptual understanding and problem-solving skills.

In conclusion, didactical phenomenology is a theoretical framework that has been widely used in RME for understanding how students interact with mathematical concepts. This approach emphasizes the importance of the subjective dimension of mathematical learning and uses student-centered tasks to support students' construction of mathematical knowledge. Research has shown that using didactical phenomenology in mathematics instruction can have significant positive effects on students' learning outcomes.

2.6.3. Self-Developed or Emergent Models

Self-Developed or Emergent Models refer to a theoretical framework that emphasizes the importance of students' self-generated understanding of scientific concepts. In this approach, students are encouraged to develop their own models of scientific phenomena through experimentation, observation, and reflection. The aim is to help students build a deep understanding of science concepts by allowing them to construct their own meaning through inquiry and exploration (Driver & Erickson, 1983).

One of the key principles of self-developed or emergent models is that scientific knowledge is constructed by the learner, rather than transmitted or imparted by the teacher. This means that teachers need to provide opportunities for students to build their own knowledge, rather than simply presenting them with pre-determined models of scientific phenomena. Through this approach, students can develop a deep understanding of scientific concepts that is meaningful and relevant to their own experiences (Driver & Oldham, 1986).

Research has shown that using self-developed or emergent models in science education can have significant benefits for students' learning outcomes. For example, a study by Wiser and Smith (2008) found that when students engaged in self-directed inquiry, they developed a more sophisticated understanding of scientific concepts and were better able to apply this knowledge to new situations. Similarly, a study by Treagust and Chittleborough (2007) found that when students were encouraged to develop their own models of science phenomena, they were better able to understand complex concepts and had increased engagement with the learning process.

Self-developed or emergent models have been found to be effective in teaching science concepts to learners of different ages and abilities. For instance, a study conducted by Osborne et al. (1993) investigated the effectiveness of teaching science through the self-developed model approach to students in grades 11 and 12 in England. The results showed that the approach improved students' understanding of science concepts and led to better retention of the information compared to other teaching methods.

Another study by Gunstone et al. (1990) reported similar findings. They used self-developed models to teach science to secondary school students in Australia, and found that the approach led to improved student engagement, increased motivation to learn, and greater conceptual understanding of the science topics.

The self-developed model approach has also been applied in other fields, including mathematics education. A study by Van de Walle and Lovin (2006) investigated the use of the approach in teaching mathematical concepts to primary school students. The results showed that the approach led to increased student engagement, improved problem-solving skills, and better understanding of mathematical concepts.

The concept of self-developed models or emergent models is the third fundamental principle underlying the instructional design of RME (Gravemeijer 1994, 1999). This concept helps close the gap between informal and formal knowledge by playing a significant role in bridging the gap between the two. It suggests that we should allow the students to utilize and create their own models when they are working on solutions to the challenges they have been given. In the beginning, the students will work together to construct a model that is already recognizable to them. After completing the generalization and formalization steps, the model will eventually develop into an independent entity. Gravemeijer (1994) refers to this process as a shift from the model-of perspective to the model-for perspective. Upon the completion of the transformation, the model can be utilized as a model for mathematical reasoning (Treffers, 1991a; Gravemeijer, 1994, 1999). The use of models in three different approaches for mathematics learning is presented in Figure 2.14.

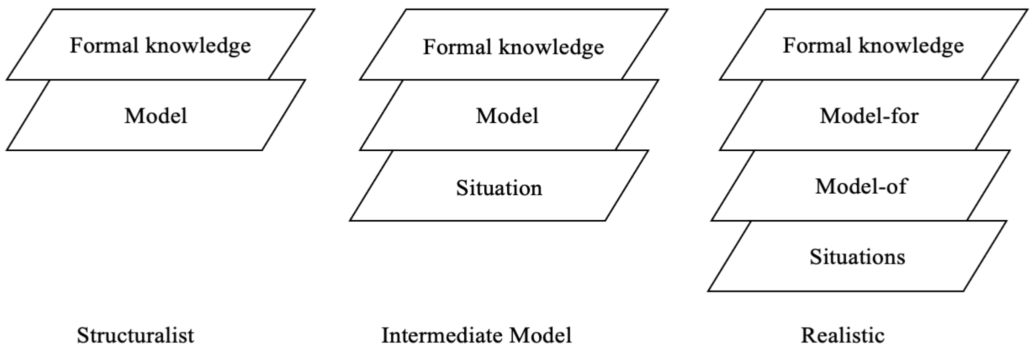


Figure 2.14. The use of models in three different approaches (Gravemeijer, 1994, p. 101)

This section began with the introduction of the phrase emergent models. Gravemeijer (1999) uses this phrase to describe the nature of the transition from model-of to model-for. As students address a contextual problem, an RME model arises from the informal answers they generate. The concept is first employed to assist informal strategies correlating to context solution strategies. As students are exposed to comparable problem-solving techniques, the selection of a method is no longer dependent on its relation to the problem situation but is highly affected by the mathematical properties of the problem. Here, the significance of the model starts to evolve as it takes on a more general aspect. At the completion of the reification procedure, the model becomes an independent entity. Gravemeijer (1999) contends that at this level, the model is more significant as a foundation for mathematical thinking than as a means of representing a contextual situation.

In conclusion, self-developed or emergent models offer a valuable framework for science education. This approach emphasizes the importance of student-directed learning and provides opportunities for students to construct their own meaning of scientific concepts. Research has shown that when used effectively, this approach can lead to significant improvements in students' learning outcomes.

2.7. Principles of RME's Teaching and Learning

The significance of RME's guiding principles for instructional design was covered in the previous section. Assuming that we have developed learning activities in accordance with the RME approach, the following questions arise: how should teaching and learning be carried out?; how should the process of utilizing these learning activities be carried out; how should teachers deliver the learning activities in the classrooms; and how are students intended to learn from the learning activities? In relation to these problems, Treffers (1991a) suggests five learning and teaching principles, including (1) constructing and concretizing, (2) levels and models, (3) reflection and special assignments, (4) social context and interaction, and (5) structuring and interweaving. These principles of teaching and learning are similar to five concepts that were discussed by De Lange (1987), namely (1) the use of contextual problem; (2) the use of models and symbols; (3) the use of student's own contribution or creation; (4) interactive learning process; and (5) intertwining of mathematics units or strands. In the sections that follow, we will go through each of the RME's teaching and learning principles.

1. Use of contextual problem (constructing and concretizing)

RME's first learning principle is that learning mathematics is a constructive activity, which runs counter to the assumption that learning is just the process of absorbing information that is provided or communicated (Treffers, 1991a). A concrete situation should serve as the initial starting point. Put another way, the instruction needs to emphasize the phenomenological investigation (Gravemeijer, 1994). Beginning with situations that require organization as a starting point, teachers can encourage student participation in manipulating multiple organizational means. In this study, we employed the contexts of Running for Fun, and Exploring Playgrounds and Blacktop Areas to develop the multiplication of fractions. Multiplication of fraction by natural number was conceptualized in the context of Running for Fun, Training for Next Year's Marahon, and Comparing the Cost of Blacktopping; whereas multiplication of fraction-by-fraction through an array model was conceptualized in the context of Exploring Playgrounds and Blacktop Areas.

2. Use of models and symbols (levels and models)

According to this guiding principle, the acquisition of a mathematical idea or skill is regarded as a process that frequently extends over an extended period of time, and that progresses through multiple levels of abstraction (moving from the intuitive level to the level of subject-matter systematics as it moves from the informal level to the formal level) (Treffers, 1991a). Now the question is, what can be done to assist in bridging the gap between different levels? We will use the phrase bridging when referring to vertical instruments. Gravemeijer (1994) proposes that educators should pay substantial attention to the visualizations, model contexts, and conceptual models that emerge from students' participation in problem-solving activities. Doing so will assist students in progressing through the various levels. In this study, students' drawings of running paths, playgrounds, and blacktop areas, and students' findings on the relations among circuit, minutes and rates, served as a bridge to the more abstract number line, array model, and ratio table.

3. Use of student's own contribution (reflection and special assignments)

RME's third learning principle focuses on elevating the learning process to a higher degree, and it is called "increasing the level of the learning process." Since the process of increasing is said to be fostered by reflection, as stated by Graveimeijer (1994) and Treffers (1991a), it is imperative that careful attention be paid to a students' own contributions and creations. Regarding the pedagogical principle, the students need to be given the opportunity and encouraged at significant points throughout the coursework to think back on aspects of learning that they have already been exposed to and to look forward to what is still to come. This must be done on a continuous basis (Treffers, 1991a). In order to put this theory into practice, we need to provide students with unique projects, including those that require them to resolve a conflict or those that are designed to encourage the students' creative productions. In this study, the number line, array model, and ratio table will be constructed based on students' problem-solving strategies.

4. Interactive learning process (social context and interaction)

According to Treffers (1991a), learning is not an individual activity but rather takes place in a social setting and is guided and stimulated by that setting. This idea is central to the fourth principle of RME's teaching and learning, which emphasizes the significance of social context. Students can learn from one another by discussing and debating ideas in groups. Based on this principle, it follows that learning mathematics should always involve participation. This indicates that actively participating in learning is crucial through activities such as negotiation, intervention, cooperation, and evaluation (Gravemeijer) (1994). Through interactivity and discussions, students will have the opportunity to present their solutions to contextual problems. It does not imply that every student follows the same educational path. Each person is unique and pursues their own educational path. Therefore, the problem should be stated so that it can be solved by individuals with varying degrees of comprehension. In addition, the instructor plays a crucial role in guiding students toward a more formal understanding of mathematics.

5. *Intertwining of mathematics units or strands (structuring and interweaving)*

The first RME's teaching and learning principle is linked with this last one. According to Treffers (1991a), learning mathematics does not comprise acquiring a series of knowledge and skills that are unrelated to one another. Instead, learning mathematics involves the construction of knowledge and skills into an organized structure. In addition, Gravemeijer (1994) states that the holistic approach implies that there are no separate modules or strands for mathematics. Instead, an intertwining of learning strands is utilized in the problem-solving process, and thus it is integrated into other themes or even topics. These arguments lead to the pedagogical principle, which states that the different learning strands in mathematics must be interwoven. In this study, the ideas of fractions are intertwined with ratio and proportion, measuring, decimal numbers, and the ability to perform fundamental operations.

Another important principle of RME's teaching and learning is visualization. Visualization refers to the use of diagrams, models, and other visual representations to help students understand mathematical concepts. Visualization can help students to make connections between concrete and abstract mathematical ideas and support problem-solving. According to Drijvers et al. (2010), visualization enables students to develop manifold representations of mathematical concepts, which can help to increase their mathematical proficiency.

In addition to visualization, the principle of multiple solution paths is another key aspect of RME's teaching and learning. Multiple solution paths refer to the idea that there are different ways to solve a mathematical problem, and students should be encouraged to explore and compare these different methods. This approach can help students to develop flexible thinking and deepen their understanding of mathematical concepts. According to Van Galen and Van Eerde (2012), multiple solution paths can encourage students to develop their own mathematical strategies and approaches.

Finally, RME's teaching and learning emphasizes the importance of reflection. Reflection involves thinking about one's own learning and understanding, and it can help students to identify their own strengths and weaknesses and monitor their own progress. According to Dolk et al. (2016), reflection can help students to gain a deeper understanding of mathematical concepts and to apply their knowledge in new situations.

In conclusion, RME's teaching and learning principles provide a robust framework for developing mathematical proficiency in students. By emphasizing problem-posing, guided reinvention, the use of contexts, collaborative learning, visualization, multiple solution paths, and reflection, RME enables students to construct their own mathematical understanding and develop a deeper appreciation of mathematics.

2.8. Multiplication of Fractions using Realistic and Mathematics Education

The multiplication of fractions is a crucial topic in mathematics education that is challenging for many students to master. In recent years, a considerable amount of research has been conducted to explore the teaching and learning of this topic. While the field has made significant progress towards improving students' learning of fraction multiplication, there is still a long way to go to ensure that all students can develop a deep understanding of the concept.

Studies have identified several obstacles that hinder students' learning of fraction multiplication, such as difficulty in grasping the abstract nature of fractions, confusion in understanding the meaning of multiplication, and limited knowledge of the properties of fractions. Additionally, previous studies have highlighted the importance of an adequate understanding of fraction division, which lays the foundation for the understanding of fraction multiplication (e.g., Gruszczyńska & Sokolowska, 2017).

Teachers need to identify these challenges and develop effective teaching strategies to overcome them.

Besides, research on fraction multiplication has highlighted the importance of providing students with multiple representations or visual models of fractions to support their understanding of the concept. For instance, researchers have found that using area models or arrays can help students visualize the multiplication of two fractions as the area of the rectangle formed by the two fractions' sides (Pan et al., 2017). Similarly, the use of number lines, especially when combined with a repeated-addition strategy, has been shown to be effective in helping students understand the distributive property of multiplication (Kim & Kim, 2019). These different representations or visual models of fractions can help students develop a more robust and flexible understanding of fraction multiplication and its properties.

RME is a student-centered approach that encourages students to develop their problem-solving skills by engaging them in real-life situations. One of the key principles of RME is contextualization. RME provides students with a context in which they can apply and practice mathematical concepts. Fraction multiplication is one of the topics that can benefit from the contextualization approach. When students are presented with problems that are familiar to them, they are more likely to develop a deeper understanding of the mathematical concepts involved.

The use of real-world contexts in fraction multiplication has been shown to be effective in improving students' understanding of the concept. For example, researchers have used real-world proportions to teach students about fraction multiplication. One study used baking recipes to teach fraction multiplication to students. Students learned how to multiply fractions by using baking recipes to determine the amount of each ingredient needed to make different sizes of muffin batches (Kucuk et al., 2019). The use of the baking recipes inspired students to learn fraction multiplication because the context was engaging, and it seemed useful in their everyday life.

The effectiveness of RME in teaching fraction multiplication has been demonstrated in different studies. For example, researchers found that integrating RME in the curriculum can improve students' comprehension of the topic and their ability to solve related problems (e.g., Van Galen & Elia, 2016; Verschaffel et al., 2018). Moreover, RME has proven to be effective in reducing arithmetic anxiety among primary school students (Ernest & Atteberry, 2018). These findings suggest that RME can be a valuable approach in teaching fraction multiplication. However, research on RME and fraction multiplication has also found that teachers need to receive adequate training and support in implementing the approach effectively (e.g., Van den Heuvel-Panhuizen, 2010).

As explained earlier, RME is an approach to teaching mathematics that focuses on developing students' ability to reason, problem-solve, and mathematize. Hence, using RME principles in instruction can support students in mathematizing their learning of fraction multiplication. Mathematizing processes are fundamental to supporting students in building their understanding of mathematical concepts, such as multiplication of fractions. Utilizing multiple models, such as number lines, ratio tables, and area/arrays, can facilitate students in developing their conceptual understanding of fraction multiplication. The following studies illustrate the effectiveness of utilizing RME in teaching fraction multiplication and studies that highlight the use of these models in mathematizing processes in learning multiplication of fractions.

Shanty et al. (2011) conducted a design research study to investigate the progress of Indonesian fifth-grade students learning the multiplication of fractions with natural numbers. They used RME as a theoretical framework and designed an instructional sequence that incorporated the key principles of problem-posing, guided reinvention, the use of contexts, and visualization. The study involved 32 fifth-grade students from two different classes in a public elementary school in Indonesia. The students were taught using the RME instructional sequence over four weeks, with pre- and post-tests administered to measure their learning progress. The researchers used a combination of qualitative and quantitative data collection methods, including classroom observations, interviews, and written assessments.

The results of the study indicated that the RME instructional sequence was effective in promoting students' learning of multiplication of fractions with natural numbers. The students showed substantial improvement in their understanding of the concepts and were able to solve more complex problems by the end of the study. The study also revealed the important role of teachers in facilitating students' learning and encouraging them to ask questions and explore their ideas. Overall, Shanty et al.'s (2011) research provided important insights into the effectiveness of RME in promoting students' learning of mathematics concepts. The study highlighted the need for more research in this area, particularly in the context of developing countries like Indonesia, where there is a need to improve students' mathematical knowledge and skills. The study also underscored the importance of designing and implementing effective instructional sequences that incorporate the key principles of RME to support students' mathematical learning.

A study by van den Heuvel-Panhuizen and Drijvers (2014) examined the effect of visual models on the teaching of multiplication of fractions using the principles of RME. The study included 82 students from three different classes in a Dutch primary school, and the students were taught using both number line and ratio table models over the course of eight weeks. The results of the study indicated that using these visual models in teaching multiplication of fractions was effective in helping students understand the concept and to mentally manipulate fractions. The researchers found that the use of the number line model was particularly useful for students who had difficulty understanding the concept of multiplication of fractions. The ratio table model, on the other hand, was found to be more effective for advanced students who were already familiar with the concept. The visual models also supported students in making connections between different operations, such as addition and multiplication, and in generalizing their strategies. The researchers found that students were able to transfer their knowledge of the visual models to solve more complex problems and to apply the same strategies to other mathematical concepts.

Overall, van den Heuvel-Panhuizen and Drijvers' (2014) study provides strong evidence for the effectiveness of using visual models in teaching multiplication of fractions using RME principles. The study also highlights the importance of using multiple models and strategies to support a diverse range of learners. By incorporating visual models into their teaching, teachers can help students to develop a deeper understanding of mathematical concepts and to become proficient problem-solvers..

Kuma et al. (2018) conducted a study to investigate the effectiveness of using a ratio table and area models in teaching multiplication of fractions using RME principles. The study involved 36 fourth-grade students from a public school in Japan, and the students were taught using either the ratio table or area models or a combination of both over the course of four weeks. The results of the study indicated that both the ratio table and area models were effective in helping students understand the concept of multiplication of fractions, identify equivalent fractions, and make connections between different mathematical operations. Students were also able to apply their knowledge of the visual models to solve more complex problems and to transfer their learning to other mathematical contexts. The study highlighted the importance of using multiple visual models in teaching mathematical concepts. By using a range of models, teachers can support a diverse range of learners and help students develop a deeper understanding of mathematical concepts. The study also emphasizes the need to incorporate RME principles into instructional design, including problem-posing, guided reinvention, and the use of contexts, to support students' learning.

Overall, Kuma et al. (2018) study provides valuable insights into the effectiveness of using visual models in teaching multiplication of fractions using RME principles. The study reinforces the importance of incorporating multiple strategies and models into instructional design, particularly in the context of developing students' mathematical understanding and skills.

Additionally, Ongadi and Adera (2019) conducted a study to investigate the use of an array model in teaching multiplication of fractions using RME principles. The study involved 36 sixth-grade students from a public school in Kenya, and the students were taught using the array model over the course of four weeks. The results of the study indicated that using the array model was effective in helping students visualize the multiplication of fractions and to identify patterns and relationships between different fractions. The visual representation of fractions allowed students to develop a deeper understanding of the concept of multiplication of fractions and to apply their knowledge to solve more complex problems. The study also found that the use of the array model supported students' understanding of multiple mathematical operations. Students were able to make connections between multiplication and division, as well as between fractions and decimals. The array model also enabled students to identify equivalent fractions and to relate fractions to naturalnumbers.

Overall, Ongadi and Adera’s (2019) study provides compelling evidence for the effectiveness of using an array model in teaching multiplication of fractions using RME principles. The study highlights the importance of providing students with visual representations of mathematical concepts and of using multiple models to support students' learning. By incorporating RME principles into instructional design and utilizing visual models such as the array model, teachers can help students develop a deep understanding of mathematical concepts and become proficient problem-solvers.

Afifah and Nurniati (2021) conducted a study to investigate the use of RME in teaching fraction multiplication to fifth-grade students. The study involved 30 students from a public elementary school in Indonesia, and the students were taught using instructional activities designed based on RME over the course of four weeks. The results of the study indicated that the use of RME facilitated students’ learning of fraction multiplication and empowered them to mathematize their learning. Specifically, students were able to connect their prior knowledge to new concepts, formulate their strategies, and engage in productive discussions about their mathematical reasoning. The study found that the RME approach enabled students to make connections between multiplication as repeated addition and multiplication of fractions. Instructional activities designed based on RME also allowed students to approach fraction multiplication as a problem-solving activity, where they were encouraged to explore different strategies, formulate their own algorithms, and explain their reasoning. Moreover, the study highlighted that the RME approach enabled students to engage in productive discussions about their mathematical reasoning. This allowed students to share their ideas, listen to others, and learn from each other's strategies.

Overall, Afifah and Nurniati's (2021) study provides evidence for the effectiveness of RME in teaching fraction multiplication to fifth-grade students. The study supports the importance of engaging students in problem-solving activities and empowering them to mathematize their learning. By incorporating RME principles into instructional design, teachers can facilitate students' learning of mathematical concepts and promote their critical thinking and reasoning skills..

Similarly, a study by De Boo and Van Langen (2014) examined the effectiveness of teaching fraction multiplication using RME principles. The study involved 61 fifth-grade students from a primary school in the Netherlands, and the students were taught using the RME approach over the course of six weeks. The results of the study indicated that teaching fraction multiplication using the RME approach improved students' understanding of the concept of fraction multiplication. The RME approach enabled students to visualize the concept of multiplication of fractions and to connect it to their prior knowledge of multiplication of natural numbers. The study found that teaching fraction multiplication using RME principles enabled students to mathematize their thinking. That is, they were able to recognize patterns and relationships between fractions, and to understand the rules governing fraction multiplication, such as the fact that multiplying by a fraction less than 1 results in a smaller product. Additionally, the RME approach enabled students to recognize the relationship between multiplication and division of fractions. This allowed them to use division as a strategy to solve fraction multiplication problems. Students were also able to apply their knowledge of fraction multiplication to solve real-world problems, demonstrating a deeper understanding of the concept. Overall, De Boo and Van Langen's (2014) study supports the effectiveness of RME principles in teaching fraction multiplication. The study highlighted the importance of visualization and connection to prior knowledge in promoting students' understanding of mathematical concepts. The RME approach can be a valuable tool for teachers in helping students develop a deeper conceptual understanding of mathematics.

A study by Trahan, Flores, and Oyadomari-Chun (2019) implemented RME principles in teaching sixth-grade students fraction multiplication using area models. The study involved 26 students and lasted for two weeks. The results of the study indicated that teaching fraction multiplication using area models facilitated students' understanding of the concept. Area models allowed the students to visualize the concept of fraction multiplication and see how it relates to their prior knowledge of multiplication of natural numbers. The study found that the use of area models enabled students to mathematize their thinking and recognize patterns and relationships between fractions. By breaking the fractions into smaller units, students were able to develop mental representation of fractions and to reason about fraction multiplication easily. Furthermore, the study highlighted the importance of building connections between mathematical concepts and real-life situations. Students were able to apply their knowledge of fraction multiplication to solve real-world problems, such as determining the area of a rectangle or the amount of ingredients needed to cook a recipe.

Overall, Trahan, Flores, and Oyadomari-Chun's (2019) study supports the effectiveness of RME principles and area models in teaching fraction multiplication to grade six students. The study emphasizes the importance of visual representation, connection to prior knowledge, and application to real-life situations in promoting students' understanding of mathematical concepts. Teachers can incorporate area models into their instructional design to deepen students' knowledge, promote critical thinking, and facilitate their ability to apply mathematical concepts to real-life problems..

Blume and Heck (2019) conducted a study to investigate how visual models, such as the number line, ratio table, and array model, can support fifth-grade students' understanding of multiplication of fractions. The study involved 31 students and lasted for four weeks. The results of the study indicated that the use of visual models supported students in constructing a more robust understanding of fraction multiplication. The visual models allowed students to see the relationship between the fractions, helping them to develop a deeper understanding of the concept. The study found that the number line model was particularly effective in supporting students' understanding of fraction multiplication. The number line allowed students to visualize the relationship between the fractions and connect it to their prior knowledge of multiplication of natural numbers. The ratio table and array models were also found to be effective in promoting students' understanding of fraction multiplication. These models allowed students to see the relationship between the fractions, helping them to develop a more concrete understanding of the concept. Furthermore, the study highlighted the importance of connecting mathematical concepts to real-life situations. Students were able to apply their knowledge of fraction multiplication to real-life problems, such as determining the amount of ingredients needed to make a recipe or the distance traveled by a car.

Overall, Blume and Heck's (2019) study supported the effectiveness of visual models, such as the number line, ratio table, and array model, in promoting students' understanding of fraction multiplication. The study demonstrates the importance of visualization and connection to real-life situations in promoting students' understanding of mathematical concepts. Teachers can use these visual models in their instruction to deepen students' understanding and promote their ability to apply mathematical concepts to real-life problems.

Similarly, another study by Clements, Sarama, and Delgado (2015) investigated the effectiveness of using area models, ratio tables, and visual fraction representations in teaching fraction multiplication to third-grade students. The study involved 240 students and lasted for 10 lessons. The results of the study indicated that using these models was beneficial for students in mathematizing their learning and building their conceptual understanding of fraction multiplication. The area model, in particular, was found to be most effective in promoting students' understanding of the concept. The area model allowed students to see the relationship between the fractions and visualize how the product was formed through multiplying the fractions' areas. The use of the area model helped students to develop a mental representation of fractions, which led to their conceptual understanding of fraction multiplication. The study also found that the ratio table and visual fraction models were effective in supporting students' understanding of fraction multiplication by breaking down the fractions into smaller and more manageable units. These models enabled students to identify patterns, which helped them to mathematize their learning and develop fluency in multiplying fractions. Furthermore, the study highlighted the importance of providing students with opportunities to connect mathematical concepts to real-life situations. Students were able to apply their knowledge of fraction multiplication to solve real-world problems, such as calculating the length of a garden fence or the amount of fabric needed to make a dress.

Overall, Clements, Sarama, and Delgado's (2015) study supports the effectiveness of using area models, ratio tables, and visual fraction representations in teaching fraction multiplication to third-grade students. The study demonstrates the importance of visualization, breaking down fractions into smaller units, and connection to real-life situations in facilitating students' understanding of mathematical concepts. Teachers can use these models in their instructional design to deepen students' conceptual understanding of fraction multiplication and promote their ability to apply mathematical concepts to real-life situations..

Furthermore, in a study by Nguyen and O'Brien (2019), they investigated the effectiveness of using the number line model to teach fraction multiplication to seventh- and eighth-grade students. The study involved 28 students and lasted for five weeks. In this study, the students used the number line to develop their understanding of fraction multiplication, and their learning was scaffolded through guided practice and direct instruction. The authors found that the number line model supported students in visualizing the relative sizes of fractions and helped them to mathematize their thinking and make better sense of the multiplication of fractions. The number line allowed students to see the relationship between the fractions and develop a deeper understanding of the concept. The authors found that the number line model helped students to visualize the fractions as distances along a line, and understand that multiplying a fraction by another fraction is equivalent to finding a proportionate distance along the line. Furthermore, the guided practice and direct instruction helped students to develop fluency in using the number line model to multiply fractions. Through the scaffolding, students were able to identify patterns and develop strategies in using the number line to solve fraction multiplication problems.

Overall, the study by Nguyen and O'Brien (2019) supported the effectiveness of using the number line model to teach fraction multiplication. The study demonstrated the importance of visualization and scaffolding in supporting students' understanding of mathematical concepts and promoting their ability to apply mathematical concepts to solve problems. Teachers can use this model in their instructional design to deepen students' understanding of fraction multiplication and support their ability to develop fluency in applying the concept.

In addition, Işıksal and Çakıroğlu (2011) conducted research on prospective teachers to explore effective strategies for helping students overcome misconceptions and difficulties related to the multiplication of fractions. The researchers proposed several strategies to support students in understanding the concept rather than simply memorizing the algorithm. One of the suggested strategies is the use of multiple representations, such as verbal expressions, figures, and graphics. By presenting the concept in different forms, students could develop a holistic understanding of multiplication of fractions. Furthermore, employing various teaching methods, particularly problem-solving approaches, could enhance students' comprehension of the topic. The importance of practice and getting students to express their reasoning was also emphasized. By encouraging students to explain their thought processes, teachers could identify any misconceptions and provide targeted interventions. Additionally, the researchers recommended the use of manipulatives and other instructional materials to create visual representations of the concept. By visualizing the operations with the help of these materials, students were more likely to comprehend the logic behind the multiplication of fractions. This approach discouraged mere memorization of rules and promoted meaningful application of the concept.

The prospective teachers in their study also suggested employing multiple strategies to help students understand the meaning of concepts before engaging in procedural problem-solving. For example, concrete materials and examples from daily life can be used to facilitate visualization and develop a deeper understanding of the concept. Problem-solving strategies can be employed to familiarize students with the concepts in a practical way. These activities are intended to encourage students to truly comprehend the multiplication of fractions instead of relying on rote memorization of rules. This approach aligns with the first RME’s first learning principle, which emphasizes the utilization of contextual problems as a foundation for student learning. By utilizing real-world situations as a starting point, students are encouraged to actively participate in the learning process (Gravemeijer, 1999).

RME emphasizes the importance of presenting mathematics in a meaningful and relatable way to students. Instead of focusing solely on abstract concepts, this approach integrates mathematics within a realistic context, allowing students to see the relevance and practical applications of the subject (Freudenthal, 1991). By presenting problems that students can relate to and find interest in, RME aims to foster active engagement and motivate students to explore and develop mathematical thinking and problem-solving skills (Van den Heuvel-Panhuizen, 2003).

Furthermore, the use of contextual problems as a starting point for learning encourages students to connect their prior knowledge and experiences to new mathematical concepts. This approach cultivates a deeper understanding and promotes a constructivist learning environment, where students actively construct their knowledge by making connections between mathematical ideas and real-world situations (Verschaffel et al., 2000).

Research supports the effectiveness of the use of contextual problems as a teaching strategy in mathematics education. Studies have shown that when students are presented with real-world or authentic problems, they are more motivated, engaged, and have a better overall understanding of mathematical concepts (Lobato, 2003; Stylianides & Ball, 2008).

In conclusion, research findings suggest that utilizing visual models, such as number lines, ratio tables, and area/arrays, in teaching multiplication of fractions can support students in mathematizing their learning and building their conceptual understanding of the operation. These models can enable students to visualize and think mathematically as they work with fractions, making connections to real-life situations beyond their mathematics class. Additionally, utilizing RME principles in instruction can empower students to connect their prior knowledge to new concepts, engage in productive discussions, formulate their strategies, and make meaningful connections between mathematics and the real world.

2.9. Fraction Learning in the Common Core Standard Initiative for Grade 5 Students

One of the key areas of focus in the Common Core State Standards Initiative (CCSSI) for grade 5 students is fraction learning. The standards aim to help students develop a deep understanding of what fractions are, how they are represented, and how they can be used in a variety of real-world contexts (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Fraction learning is a critical component of the overall mathematics curriculum, as it lays the foundation for more complex mathematical concepts in higher grades.

The CCSSI for grade 5 students includes a variety of specific standards related to fraction learning. For example, students are expected to understand the concept of equivalent fractions, as well as how to add, subtract, multiply, and divide fractions (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In addition, students are expected to be able to apply their understanding of fractions to solve real-world problems, such as those involving measurements, ratios, and rates (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

To help students achieve these goals, the CCSSI includes a variety of instructional strategies and materials for fraction learning in grade 5. For example, teachers are encouraged to use visual representations and manipulatives to help students understand the concept of fractions as parts of a whole (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In addition, students are expected to use mathematical reasoning and communication skills to explain their thinking and justify their answers when solving fraction problems (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

According to the Common Core Standards Initiative (NGA, 2016), fractions have been taught since the third grade, during which time students have developed their understanding of fractions as numbers, such as understanding a fraction as a part of a whole, representing fractions as numbers on a number line, explaining fraction equivalence, and comparing fractions by reasoning. In fourth grade, students built fractions from unit fractions, mastered the decimal notation for fractions, and compared decimal fractions by using and expanding their prior knowledge of operations on natural numbers.

As the focus of this study was learning fractions in fifth grade, the fraction multiplication standards for fifth grade were included in Table 2.2 under the 'Number and Operations of Fractions' domain.

Table 2.2. Standards under the 'Number and Operations of Fractions' domain in grade 5 (NCTM, 2022, pp. 36-37)

Standard	Sub-Standard
Apply and extend previous understanding of multiplication and division to multiply fractions.	1. Interpret a fraction as division of the numerator by the denominator $\left(\frac{a}{b} = a \div b\right)$. Solve word problems involving division of natural numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For instance, interpret $\frac{1}{4}$ as the result of dividing 1 by 4, noting that $\frac{1}{4}$ multiplied by 4 equals 1, and that when 1 whole is shared equally among 4 people each person has a share of size $\frac{1}{4}$.
	2. Apply and extend previous understandings of multiplication to multiply fractions or natural number by a fraction. a. Interpret the product $\left(\frac{a}{b}\right) \times q$ as a part of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. For instance, use a visual fraction model to show $\left(\frac{2}{3}\right) \times 4 = \frac{8}{3}$, and create a story context for this equation. Do the same with $\left(\frac{2}{3}\right) \times \left(\frac{4}{5}\right) = \frac{8}{15}$. In general, $\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{ac}{bd}$. b. Find the area of a rectangle with fractional side length by tilling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.
	3. Interpret multiplication as scaling (resizing), by: a. Comparing size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. b. Explaining why multiplying a given number by a fraction greater than 1 result in a product greater than the given number (recognizing multiplication by natural numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results

	in a product smaller than the given number; and relating the principle of fraction equivalence $\frac{a}{b} = \frac{(n \times a)}{(n \times b)}$ to the effect of multiplying $\frac{a}{b}$ by 1.
	4. Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

2.10. Conclusion

The mathematizing process is crucial for developing formal mathematics. The RME theory acknowledges the significance of contextual limitations in inhibiting the acquisition of abstract concepts from ordinary life experiences. RME prioritizes working within familiar contexts as a starting point for learning (Gravemeijer, 1994), such as Running for Fun and Exploring Playgrounds and Blacktop Areas, to facilitate the development of students' understanding to more abstract concepts. This theoretical approach has been shown to be helpful in teaching complex mathematical concepts effectively, including fractions.

The RME model for teaching multiplication of fractions emphasizes the use of visual models and linking mathematical symbols to these models (Bakker & Gravemeijer, 2004). This approach builds on students' existing knowledge and experiences in order to help them develop a deeper understanding of the underlying mathematical principles. The model uses the concept of sharing equally and changing the scale of the unit fraction to facilitate understanding of multiplying fractions. By drawing connections between real-life contexts and math concepts, the RME model can help to enhance students' understanding of fractions and their applications.

The CCSSI (2010) recommends the use of various models for learning fractions, which include the number line, ratio table, and area/array model. These models have been shown to be effective in helping students develop a better understanding of fractions and their relationships, as well as in facilitating computation and problem-solving. The number line, for example, is a visual representation that can help students understand the relationship between fractions and their position on the number scale. The ratio table helps students understand the concept of equivalence and the relationships between fractions (Lamon, 2012), while the area/array model can help students visualize how fractions are made up of parts of a whole.

By utilizing these models in teaching, educators can provide students with a visual and conceptual foundation for learning fractions. Additionally, hands-on activities such as cutting up objects to represent fractions can also facilitate learning by providing students with a concrete representation of the concept. The combination of visual and tactile experiences can be especially helpful for students who struggle with abstract concepts, making it easier for them to understand the underlying principles of fraction learning.

In summary, the RME model of mathematizing and the use of models such as the number line, ratio table, and area/array models are important tools that can assist teachers in enhancing students' understanding of fractions and help them develop skills such as computation and problem-solving. By incorporating real-world contexts and hands-on experiences to facilitate student understanding, educators can set the stage for long-term success in mathematics. The CCSSI's emphasis on fraction learning and recommended instructional strategies are an important step towards achieving these goals.

Additionally, the study's learning activities provided a strategy for designing a curriculum highlighting mathematics as an interconnected and closely related subject. In addition, when incorporated into a curriculum that explores mathematics from both horizontal and vertical perspectives, it can generate possibilities for all students. For instance, understanding fractions begins with real-world scenarios and subsequently creates models that combine fraction operations into mathematical computations (Graeber et al., 2018). Besides, the use of multiple representations or visual models of fractions can help students develop a deep understanding of the concept.

CHAPTER 3

METHODOLOGY

This study investigated how students' learning of fraction multiplication evolves when they engage in the learning activities designed based on the Realistic Mathematics Education (RME) approach. Specifically, the two guided research questions are:

1. How do mathematizing processes facilitate fifth-grade students' learning of multiplication of fractions using Realistic Mathematics Education-based instructional activities?
2. What obstacles do fifth-grade students encounter when learning multiplication of fractions using Realistic Mathematics Education-based instructional activities?

3.1. Design Research

Design research (Gravemeijer & Cobb, 2006) was used in this study. This type of research is also referred to as developmental research as the instructional activities are developed. According to Bakker (2019), design research in education is a critical component in which the development of new educational materials (e.g., computer instruments, learning activities, or a development program). Bakker (2019) further revealed that design and research are linked in design research: the design is based on research, and the research is based on design. Integration of the development or testing of theory into the design of learning environments is expanded. The theoretical basis and conclusions distinguish design research from studies attempting to construct instructional materials through iterative prototyping, testing, and improvement cycles. Plomp (2010, cited in Bakker, 2019) described educational design research as the systematic study of designing, developing, and evaluating educational interventions – such as teaching-learning strategies and materials, as strategies to certain challenges, to enhance our understanding of the components of these interventions and the design and development processes.

According to Cobb et al. (2003) and Phillips (2006), there are five defining aspects of design research:

1. The objective is to establish theories regarding learning and the strategies intended to facilitate learning.
2. The interventionist nature of design research implies that understanding and improving a situation are intertwined.
3. The prospective and reflective components of design research should not be separated by a teaching experiment. In putting hypothesized learning into practice, researchers compare hypotheses with actual learning as observed (reflective part). Each lesson can be followed with reflection, regardless of the length of the teaching experiment. This type of research can lead to modifications to the original lesson plan.
4. Design research cycles: iterative innovation and revision (Figure 3.1). Multiple conjectures about learning can be tested. Preparation and design, teaching experiment, and retrospective analysis are common processes. A retrospective analysis supports a new cycle. Other types of educational research build on prior experiments and researchers iteratively improve materials and theoretical concepts between studies, but with design research changes might occur during a teaching experiment or series of teaching experiments. Design researchers welcome unexpected variation to determine how resilient their ideas and designs are.

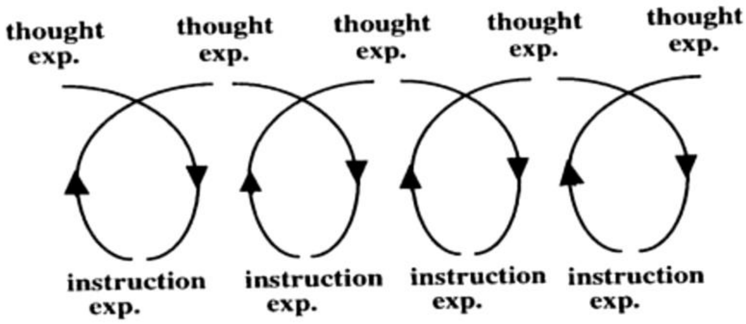


Figure 3.1. Design research as a cyclic process of thoughts and instruction experiments (Gravemeijer, 2004, p. 108)

5. Developing theory must function. Design research theory is often established for a specific domain, such as history education. It must be generic enough to be used in other classrooms. Transferability applies in such instances.

We aimed to design the research discussed here so that the findings would be useful in primary school mathematics and add to learning theories, particularly the theory of RME. We chose developmental research as an overarching research design to achieve these goals because it entails experiencing and reflecting on the developmental process. Thus, it could result in appropriate or renewed teaching programs as a continuous reflective process and modification (Freudenthal, 1991; Gravemeijer, 1994).

Specifically, for this study, we designed a Hypothetical Learning Trajectory, or HLT, for learning the multiplication of fractions. This instrument served both as a design and research instrument. More information about HLT will be further explained in section 3.2.

In this study, we followed three phases in design research (Gravemeijer, 1994; Bakker, 2004), namely:

Developing a preliminary design

The HLT was used as a guideline in preliminary design to design and develop learning activities, which are made up of three components: learning goals, instructional activities, and the tools that will be used, as well as a conjecture of students' thinking and understanding that will evolve during the implementation of instructional activities.

Conducting pilot and teaching experiments

The designed learning activities were evaluated in a pilot experiment to see if the initial learning activities were feasible. In this pilot experiment, two students were involved. The learning activities that had been adjusted based on the findings from the pilot experiment were implemented with three students involved in the teaching experiment; and revised and designed based on the implementation in the teaching experiment. The results of the teaching experiment were utilized as a basis for making adjustments to learning activities and student conjectures. Furthermore, HLT was used as a guide to focus on teaching, interviewing, and observing throughout this phase.

Carrying out a retrospective analysis

In the retrospective analysis, the data obtained from the teaching experiment was analyzed. In this phase, the HLT was compared to students' actual learning. We recorded the learning experience so that we might learn more about the subject of interest. We reviewed all the transcripts and watched the videotapes chronologically per learning moment. Conjectures about students' thinking were produced and documented using HLT and research questions as guidance. Crucial moments were discussed with colleagues to see if they agreed with our interpretation or if they could come up with an alternative (peer examination). The findings of the retrospective analysis would be used to improve the HLT and to answer the research questions.

3.2. Hypothetical Learning Trajectory (HLT)

Simon (1995) introduced HLT through a form of the Mathematics Teaching Cycle. This cycle was constructed by Simon as a method of the cycles interlinkages of aspects involving the teacher's knowledge, thoughts, and decision-making process regarding his/her preparations. Simon (1995) stated that two aspects were taken into account for the objective and sequence of the learning activity. These aspects were the teacher's mathematical knowledge and the teacher's hypotheses about the students' thoughts. Simon gives as his justification for his choice of the term "hypothesis" the fact that the teacher has no immediate links to students' thoughts but is able to deduce the pattern of the students' conjecture based on observations of the activities of the students. According to Simon (1995), it thus suggests that the teacher could, therefore, feasibly make comparisons of his/her knowledge of a certain notion from his/her construction of students' thinking, but it is impossible for the teacher to know in advance the exact thinking that the students have.

Simon (1995) provides the instructional objective of the teacher as the starting point for constructing a hypothetical learning trajectory. These considerations led to the development of this idea. Simon claims that when they use the phrase Hypothetical Learning Trajectory, they are referring to the:

"an anticipation of how the process of learning could develop. This is a hypothetical situation since the actual progression of the learning process cannot be predicted in advance. It exemplifies a pattern that is to be anticipated. The learning process for each student progresses in a unique manner, but one that is frequently analogous to other students' experiences. This is based on the assumptions that an individual's learning follows a pattern, that the classroom inhibits mathematical activity frequently in ways that are observable, and that a number of students who attend the same class could indeed advantage from performing a similar mathematical task. An HLT provides the teacher with a basis for selecting a specific instructional activity; hence, I base my design selections on my best prediction of how learning could go (Simon, 1995, p. 135)."

In order to highlight features of a teacher's knowledge that are founded on a constructivist approach and are similar to both advance design and impromptu making decisions, the phrase Hypothetical Learning Trajectory is employed (Simon, 1995). An analogy to a journey is used by Simon to explain why he decided to use the term trajectory to describe a strategy. In a nutshell, the trajectory refers to the route that one really takes, whereas the hypothetical trajectory refers to the route one had previously anticipated (Simon, 1995).

According to Simon (1995), HLT is made up of the following three components, namely:

1. Learning goal, which guides the teacher's preparation;
2. Plan that the teacher creates with the learning activities; and
3. Hypothetical learning process, which predicts how students' thinking and knowledge would develop in relation to the learning activities.

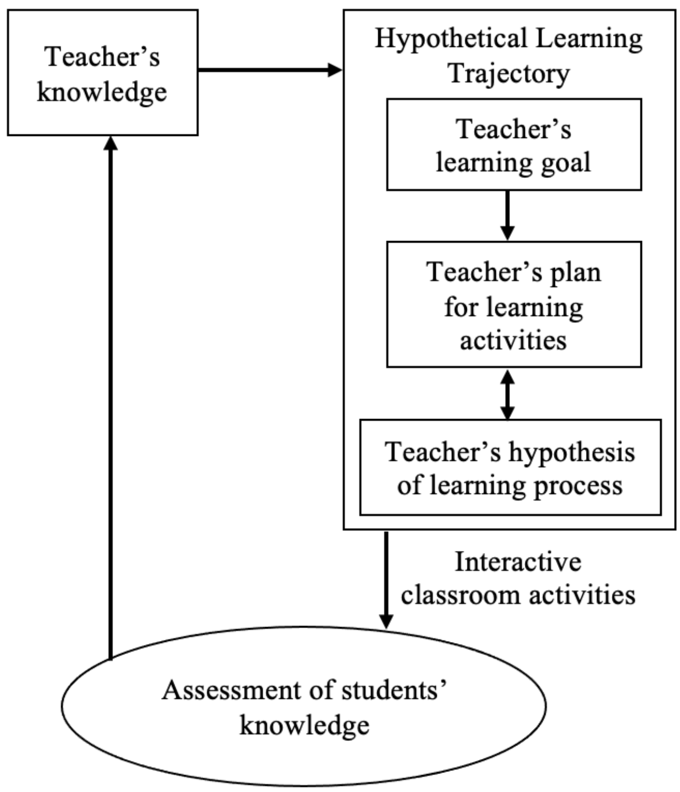


Figure 3.2. Mathematics teaching cycle (Simon, 1995, p. 136)

The teacher evaluates the objectives s/he intends to attain throughout an activity plan, and in this plan, s/he expresses his assumptions regarding the planned lesson. Simon believes that the teacher is able to select a specific learning goal for the lesson. This objective defines the circumstances and processes that the teacher will build in to generate his/her hypotheses concerning the learning process of the students. The teacher must structure the information into a structured plan in order to accomplish his/her goals.

Designing HLT requires constructing a hypothetical learning process for a certain sequence of activities once the learning goal has been determined. Simon and Tzur (2004) pose the following thought for the teacher to consider when designing a sequence of activities: “What activity, accessible to students, can serve as the foundation for them to fulfill the learning goal?”. The teacher can next examine the materials to determine which tasks afford him/her these opportunities to meet the learning goals.

Although learning objectives guide establishing the hypothesized learning trajectory, activity design and assumptions regarding the student learning process are interrelated. The activities are chosen depending on the teacher’s perceptions about the learning process, and the learning process is hypothesized according to the activities that are going to be engaged (Simon & Tzur, 2004). In this regard, several hypotheses support these theories:

1. The formulation of an HLT depends on comprehending the students' existing knowledge.
2. HLT is a tool for organizing the study of certain mathematical ideas.
3. As mathematical tasks offer instruments that facilitate the acquisition of certain mathematical ideas, they are essential to imparting mathematical knowledge to students.
4. Due to the hypothetical and naturally unpredictable character of this procedure, the teacher is continuously involved with adjusting every element of the HLT (Simon & Tzur, 2004, p. 93).

After the activities for the hypothetical learning path have been decided upon, the teacher is in a position to think about his/her theories regarding the learning process in greater depth. To do this, s/he should think about the various questions that may be raised by the students as they go through the activities. By posing these questions, the teacher is also providing students with alternative responses, allowing them to comprehend the aspects of the material that puzzle them.

Following the formulation of the learning goal(s) and the selection of the activities, the teacher will perform an assessment of the work s/he has completed and will be given the opportunity to reformulate his/her HLT (Simon, 1995).

Figure 3.2 shows that assessing students is an ongoing process that can result in adjustments being made to the teacher's knowledge, which in turn might lead to an additional or adjusted HLT.

Creating an HLT before beginning classroom teaching is a procedure by which a teacher makes a plan of activities to be carried out in the classroom by his/her students. And yet, the teacher and students together produce an interaction that is, due to the social aspect of the encounter, distinct from what the teacher had imagined. This is because the teacher interacts with and observes the students. In this setting, the teacher's beliefs on the level of knowledge obtained by the students might shift, and s/he acquires the potential to alter the HLT that s/he had previously developed (Simon, 1995).

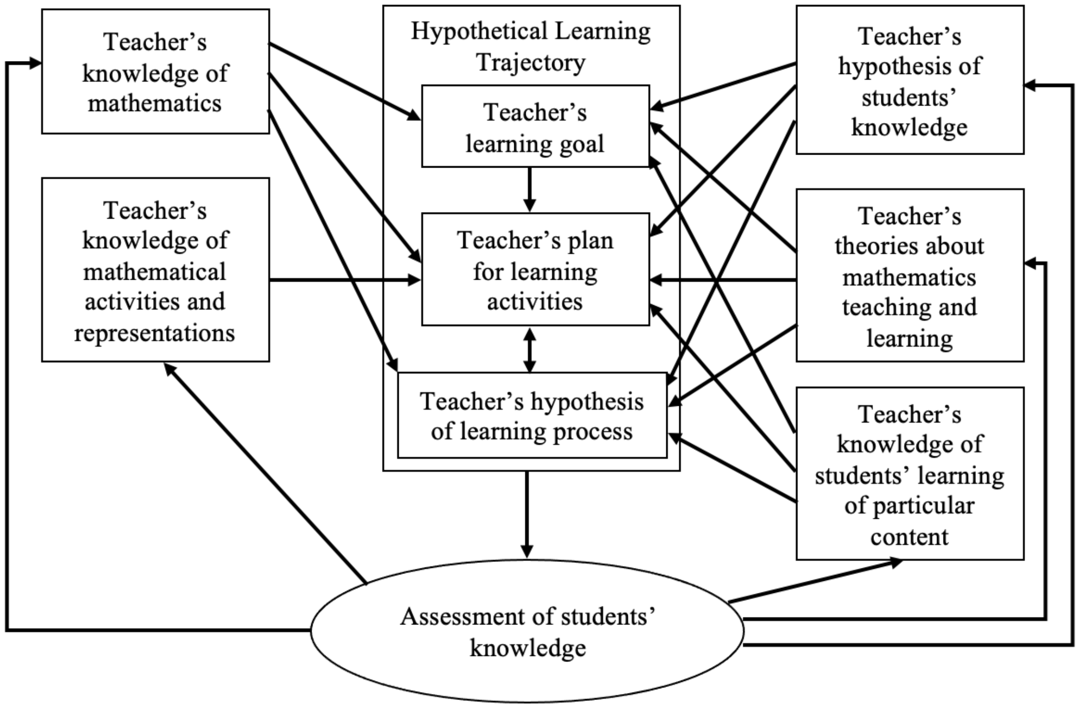


Figure 3.3. Mathematics teaching cycle (Simon, 1995, p.137)

The links between the areas of teacher knowledge, the HLT, and interactions with students are depicted in Figure 3.3. Simon (1995) provides an explanation for the figure as follows.

"Starting at the very top of the figure, the teacher's prior knowledge of mathematics, in conjunction with the teacher's beliefs about the students' prior mathematical knowledge, lead to the determination of a learning goal. The development of learning activities and a hypothetical learning process is influenced by these fields of knowledge, the learning goal, the teacher's knowledge of mathematical activities and representation, his/her knowledge of students' learning of specific topic, in addition to the teacher's interpretations of teaching and learning [...]. It is certainly not true that the adjustment of the HLT only takes place during the time spent preparing between lessons. The teacher is always involved in adjusting the learning trajectory that s/he has hypothesized in order to more accurately represent his/her improved understanding. Quite often the instruction has to be adjusted somewhat, whereas other situations the entire approach that was being taken needs to be scrapped in support of a method that is more applicable. Modification is possible to one or more of of the three components of the HLT, including the goal, the activities, and the hypothetical learning process. These modifications can be made regardless of the degree to which the components are adjusted (Simon, 1995, p. 138)."

The process of developing an HLT involves the instructor planning potential scenarios and pathways along which learning might take place in the context of specific assignments. If the results of a mathematical task do not provide the teacher with enough evidence to support his/her HLT, s/he may make changes to the task itself, or s/he may revise his/her knowledge of the students' conceptual understanding, or s/he may propose new tasks in different contexts (Simon & Tzur, 2004). One of the most challenging yet essential issues in teaching mathematics nowadays is the design of the learning trajectories of students. The opportunity to learn about students' mathematical development and how we might positively influence their mathematical development as teachers make this an issue of great interest (Steffe, 2004, p.130).

According to some research findings, the development of HLT presents an opportunity for mathematics teachers (Ivars et al., 2018; Sztajn et al., 2012; Wilson et al., 2014; Wilson et al., 2017). Sztajn et al. (2012) assert that learning trajectories, which are used in learning, provide four frameworks for teaching mathematics. These four frameworks are (i) mathematical knowledge for teaching; (ii) activity analysis; (ii) discussion as classroom activity practices; and (iv) formative assessment. They propose two frameworks to (i) construct teaching from the point of Learning Trajectories, and (ii) describe elements of instruction based on Learning Trajectories. Both of these attempts are made in the context of the Learning Trajectories framework. This second schema aims to establish connections between the constituents of HLT and the four subfields that make up "Mathematical Knowledge for Teaching" (Ball & Bass, 2003; Ball et al., 2008).

Wilson et al. (2014) reported an experiment in which the researchers investigated the design of a program designed to help the teaching-learning process on a learning trajectory by reconsidering from the point of view of mathematical knowledge for teaching. This was done to determine whether the program was successful in its intended purpose (Ball et al., 2008). According to the findings, “subject knowledge” may be taught to teachers through professional learning activities focusing on expanding their pedagogical content knowledge. These activities are included in learning trajectories.

Ivars et al. (2018) conclude that employing an HLT as a guideline to perceive students’ mathematical reasoning can enable teachers to enhance their engagement. The authors of this study had twenty-nine prospective teachers take part in a learning environment in which they were given the task of interpreting how students thought about the idea of fractions using an HLT as a basis for their interpretation. These authors believe that the enhanced perceptive skills that the teachers gained as a result of this learning activity were associated with the trainee teachers’ increased mathematical content knowledge. This learning activity helped teachers develop more detailed discussions when interpreting students’ mathematical thinking.

3.3. Hypothetical Learning Trajectory (HLT) for Learning Multiplication of Fractions

According to Gravemeijer and Cobb (2006), the purpose of the preliminary phase (phase one) of a design research experiment is to establish a local instruction theory that may be elaborated and refined during the course of executing the intended design experiment. It consists of conjectures about possible learning processes and conjectures about possible means of supporting the learning processes. In this study, conjectures of students’ thinking were hypothetized in the preliminary phase and compared with the students actual learning in pilot phase (phase 1) and experimental phase (phase 2).

3.3.1. Conjecture about Possible (Mathematical) Learning Processes and Means of Supporting the Learning Processes

Through this design research, the researcher conducted a sequence of activities that developed an understanding of multiplication of fractions. This design research emphasized the transition from algorithm mastery and memorization to problem comprehension and the discovery of students' own strategies.

In this sequence of activities, number line, ratio table, and array/area served as models. The number line was used in the first context, Running for Fun, the ratio table was used in the second context, Training for Next Year's Marathon, and the array/area model was used in the third and fourth contexts, Exploring Playground and Blacktop Areas and Comparing the Cost of Blacktopping, respectively.

The Running for Fun context was used as a starting point to connect the running route with number line as a model from measurement context. The 4th water bottle and the 6th marker were expected to be located on the same line, corresponding to a half ($\frac{1}{2}$) of the route, based on the drawing illustrating the concept of equivalent fractions. Students may have discovered additional equivalent fractions, such as $\frac{3}{12}$ (third marker) = $\frac{2}{8}$ (second water bottle) and $\frac{9}{12}$ (ninth marker) = $\frac{6}{8}$ (sixth water bottle). On the basis of the concept of equivalent fraction, it was conjectured that the students would observe that Andrew and Bella ran further than they did last year (they both reached the halfway point last year). As students worked on the problem involving determining the distance Andrew and Bella ran, they were expected to relate the measurement context to the number line and indicate fractions on the number line. The students also conjectured a double number line after combining the fractions with the length of the route. Students were expected to develop their own formula for multiplying fractions by natural numbers based on the first context.

The ratio table appeared in the Training for Next Year's Marathon context. In this problem, students were required to determine the relationships between minutes, completed circuits, and rate. It was expected that students would recognize that, in order to calculate rate, minutes must be divided by the number of circuits completed. Olivia completed only one-half of the track at a rate of 18, so it may be difficult to locate her minutes. It was hypothesized at this time that students might come up with the concept of doubling and halving. In addition, students might discover that Ethan, Benjamin, Emma, Isabella, and James all have the same rate, namely 20, and then calculate the minutes of Emma, Isabella, and James using a mathematical idea that, in order to maintain equivalence, the ratio of the related numbers must remain constant.

As students worked on Exploring Playgrounds and Blacktop Areas context, they were expected to develop an array/area model to illustrate the relationship between playgrounds and lots, blacktops and playgrounds, and blacktopped-playgrounds and lots. In addition, it was expected that the students would determine that the blacktop areas of the two gardens, Botany and Gulhane, were equivalent. Additionally, it was anticipated that the concept of equivalence versus congruence would be discussed at the math congress. Even though the areas were identical, they were not necessarily congruent depending on how the students drew them. When cutting fifths or fourths vertically (or horizontally), it was expected that students would encounter overlapping portions and struggle to determine the fractional part.

In the last context of Comparing the Cost of Blacktopping, students will investigate strategies for multiplying with equivalent fractional forms (percentages and decimals) and extend their work to rational numbers. Students were expected to extend their investigation of the commutative property of fraction multiplication to percentages. In addition, they would interchange the numerators (or denominators) of two fractions to determine their product.

The models, their imageries, and potential mathematical discourse topics in the learning activities are shown in Table 3.1.

Table 3.1. Models, imageries, and potential mathematical discourse topics of the learning activities

Model	Imagery	Activity	Potential Mathematical Discourse Topics
Number line	Running route	Running for Fun	• Fractions represent a relation. • Relation among fractions, i.e., equivalent fractions. • Multiplication of fraction with natural number.
Ratio Table	Training record	Training for Next Year's Marathon	• To maintain equivalence, the ratio of the related numbers must be kept constant. • Multiplication of fraction with natural number. • The properties (distributive, associative, and commutative) that holds for natural numbers, also apply for rational numbers.
Array/Area	Playground and blacktop representing area	Exploring Playgrounds and Blacktop Areas	• Fractions represent a relation. • The whole matters. • Multiplication of fractions.
Array/Area	Playground and blacktop representing area	Comparing the Cost of Blacktopping	• Multiplication of fractions. • The properties (distributive, associative, and commutative) that holds for natural numbers, also apply for rational numbers.

3.3.2. The Learning Activities

In these activities, the researcher encouraged students to solve problems by drawing images or using other methods that were meaningful to them. In each activity, they were asked to write down their own strategies in solving the problems. After individually working out how to solve the problem, the students participated in a so-called "math congress" to discuss their strategies.

The math congress facilitated rich observation and discussion of mathematical thinking. According to Fosnot and Dolk (2002), the purpose of math congress was to debrief the strategies that students employed, disclose multiple representations of mathematical thinking, and foster a deeper understanding of concepts. During this math congress, students shared their ways of thinking, and the researcher gave the students an opportunity to observe, hear, and understand a variety of approaches to problem solving. The learning environment evolved into a community of learners where both the researcher and the students engaged in conversation, shared their thoughts, found new approaches to solving mathematical problems, and discovered via discourse new perspectives on mathematics.

In designing the following sequence of activities, RME was used as the learning approach for students to develop their understanding of the multiplication of fractions. These learning activities were inspired by Context for Learning Mathematics (Hellman & Fosnot, 2007) and Mathematics in Context (Holt et al., 2003) books in which the contexts were adjusted to the location where the students live so they could represent the context with their real-world.

Activity 1: Running for Fun

Learning Goal

Students develop several big ideas related to multiplication with fractions.

Materials

Marathon route sheet (Worksheet 1) – one per student.

Description of the Activity

The context was about two cousins who trained together for this year's marathon, hoping to improve their distance from last year's marathon. The problem context is as follows.

> Andrew and Bella are cousins, and they trained together for this years' a 26-km marathon, hoping to improve their distance. Last year they both ran $\frac{1}{2}$ of the route. This year Bella ran $\frac{7}{12}$ of the route and Andrew ran $\frac{5}{8}$. They know this because there are markers placed every twelfth of the route's total length, to show runners where they are and how much farther they have to go to finish. There are also eight water stations equally-spaced along the way. The last one is at the finish line. Andrew and Bella want to know how many kilometers they ran this year and how much better they did than last year.

A picture of the marathon route (Figure 3.1) was be given to the students. The marathon route is from Middle East Technical University (Orta Doğu Teknik Üniversitesi) to an International School. The markers and water stations were drawn in the picture.

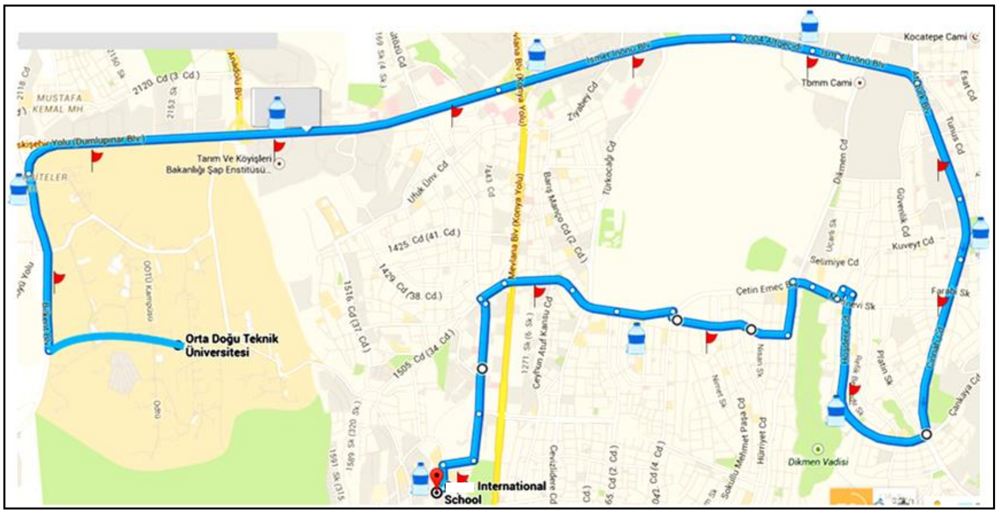


Figure 3.4. Marathon route

The researcher facilitated an initial conversation before the students began their discussions. The researcher made sure that the students understood that there are markers every twelfth of the entire length and water stations every eighth. Students were given the opportunity to share their initial opinions before being assigned the task of solving the problem.

Moreover, it was investigated whether the students realized that every third marker would be across from a water bottle station (because 4 is a common factor of 12 and 8). Would students also anticipate that every two water bottle stations will require three markers? Students would be able to investigate and discuss their thoughts with the researcher. In Activity 2, the ideas that might develop, such as the idea of multiplication of fractions as repeated addition, was discussed.

Conjecture of Students' Thinking and Expectation

As the students tried to solve the problems using their prior knowledge, there were some possible strategies which might emerge.

- To find $\frac{5}{8}$ of 26 or $\frac{5}{8} \times 26$, students would try to find $\frac{1}{8}$ of 26 ($= 3\frac{1}{4}$) first, and then multiplied $3\frac{1}{4}$ by 5 or using the idea of repeated addition in which $3\frac{1}{4}$ added five times. Students would use a similar idea to find $\frac{7}{12}$ of 26 or $\frac{7}{12} \times 26$.
- Students might use the double number line to show repeated addition, as shown in Figure 3.2 below.

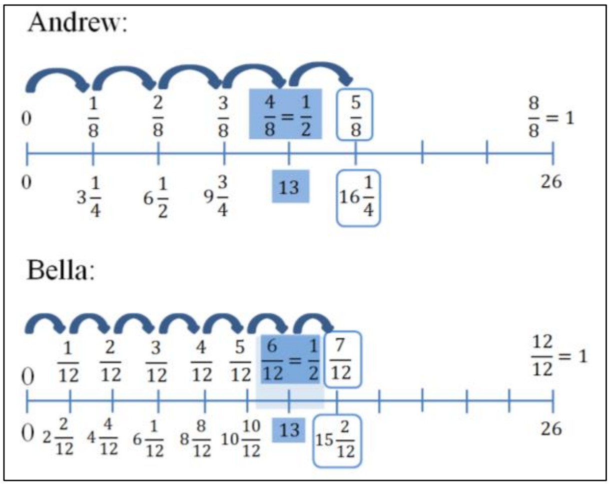


Figure 3.5. Double number line

- By using distributive strategy, students might decompose $\frac{5}{8}$ into $\frac{4}{8}+\frac{1}{8}=\frac{1}{2}+\frac{1}{8}$ and then multiplying it with 26, as explained below.

  $\frac{5}{8}\times 26=\left(\frac{1}{2}+\frac{1}{8}\right)\times 26=\left(\frac{1}{2}\times 26\right)+\left(\frac{1}{8}\times 26\right)=13+3\frac{2}{8}=13+3\frac{1}{4}=16\frac{1}{4}$

  The distributive strategy as above would also be used to calculate $\frac{7}{12}\times 26$.
- By using proportional reasoning, the students would try to find $\frac{1}{8}$ of 26 by firstly find $\frac{1}{2}$ of 26 (=13), then $\frac{1}{4}$ is $6\frac{1}{2}$, and $\frac{1}{8}$ is $3\frac{1}{4}$ (Table. 3.2).

Table 3.2. Proportional reasoning

Andrew

Fraction Distance	Kilometers
$\frac{1}{2}$ of 26	13
$\frac{1}{4}$ of 26	$6\frac{1}{2}$
$\frac{1}{8}$ of 26	$3\frac{1}{4}$
$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$	$13+3\frac{1}{4}=16\frac{1}{4}$

Bella

Fraction Distance	Kilometers
$\frac{1}{2}$ of 26	13
$\frac{1}{4}$ of 26 or same as $\frac{3}{12}$ of 26	$6\frac{1}{2}$
$\frac{1}{12}$ of 26	$2\frac{1}{6}$
$\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$	$13+2\frac{1}{6}=15\frac{1}{6}$

Activity 2: Math Congress – Running for Fun

Learning Goals

- Students share their ideas in solving problem especially in decomposing numbers and using partial products.
- Students revisit their strategies discussed in the math congress and use the idea of decomposing numbers and partial products to solve the string of Minilesson 1.

Materials

Students' answers from Activity 1 (Worksheet 1) and Worksheet 2.

Description of the Activity

As the students discussed their strategies, the discussion was structured with some considerations.

- What big ideas were most likely to be discussed?
- What were the potential sources of confusion?
- What pieces of work might elicit a response from the students that would allow them to make a generalization?

The discussion began with one of the students' works that used the repeated addition strategy (or divided the 26 by 8 and multiplied by 5). Decomposing $\frac{5}{8}$ into $\frac{1}{2}+\frac{1}{8}$, multiplying the pieces using landmark fractions, and then putting the partial product together was an example of a decomposing strategy. Students' works were utilized to discuss this. Furthermore, if any students had noted that every third, sixth, and ninth marker is located across from the second, fourth, and sixth water stations, a discussion about why this occurs would lead to the concept of fraction equivalency.

The discussion should not only be a repetition of the strategies discussed by the students. Rather, it was an opportunity to concentrate on a several big ideas. A set of related questions called Minilesson 1 (Figure 3.6) would also be presented to the students to allow them to revisit and extend the big ideas discussed in the Math Congress. The concept of fractions as an operator was introduced in this Minilesson. Students could utilize a double number line to solve the problem. The numbers in the problems were carefully chosen to urge students to decompose fractions and apply the distributive property, as decomposing $\frac{5}{8}$ into $\frac{1}{2}+\frac{1}{8}$ may appear to them.

String of related problems:

$\frac{1}{2} \times 36$

$\frac{1}{4} \times 36$

$\frac{1}{8} \times 36$

$\frac{5}{8} \times 36$

$\frac{7}{8} \times 36$

$\frac{5}{8} \times 48$

Figure 3.6. Minilesson 1: Fractions as Operators

Conjecture of Students' Thinking and Expectation

It is expected that through students' own contribution in solving problem in Activity 1, several big ideas would emerge:

- Fractions which are difficult to work (i.e., "unfriendly" fraction) with can be decomposed into unit fractions.
- Equivalent of fractions, for instance: $\frac{1}{2} = \frac{4}{8} = \frac{6}{12}$.
- Equivalent fractions could be used to build multiplication strategies for fractions. For instance: $\frac{5}{8} = 5 \times \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$.
- The whole matters. Students could not always compare the numerators or denominators while determining whether $\frac{7}{12}$ is more than $\frac{5}{8}$.
- For fractions, the distributive property holds for multiplication over addition. The partial product of $\frac{1}{2} \times 26$ and $\frac{1}{8} \times 26$ can be used to find the product of the whole, $\frac{5}{8} \times 26$.

- Through proportional reasoning, the students might determine that if $\frac{1}{8} \times 26 = 3\frac{1}{4}$, then $\frac{5}{8}$ of 26 = $5 \times 3\frac{1}{4} = 16\frac{1}{4}$.

Activity 3: Training for Next Year's Marathon

Learning Goals

- Students use landmark fractions and partial products when multiplying a fraction by a natural number.
- Students explore multiplication and division (a natural number by a fraction) and the relationship between the operations.

Materials

Worksheet 3 – one per student

Description of the Activity

In this activity, the students tried to solve the Training for Next Year's Marathon problem, the following story was delivered.

> For next year's marathon competition not only Andrew and Bella who do the training. Some members of the club are also doing the training. They train in a park that has a 3-kilometers running track with markers to indicate the part of the track that has been completed. Several of the members decide to keep the record of their result as follows.

	Minutes	Circuit of Track Completed	Rate (Minutes per Circuit)
Alex	120	4	
Ethan	60	3	
John	45	3	
Elizabeth		2	30
Benjamin		1	20

Olivia		$\frac{1}{2}$	18
Emma		$\frac{1}{4}$	20
Isabella		$\frac{3}{4}$	20
James		$1\frac{1}{2}$	20
Rafa		$2\frac{3}{4}$	30

At the beginning of the lesson, the researcher invited the student to look at the relationship between "minutes" and "circuits of track completed." Students were encouraged to draw pictures in order to determine the rate, and were guided to understand that the size of the whole matters, and in all cases, it is one circuit. Students were asked to classify the multiplication and division problems into categories. They were also asked about the connections between the multiplication and division problems.

As the students work on the problem, the researcher encouraged them to look for relationships in the data that maintain equivalence if the ratio is kept constant. In this situation, the students might discover that Ethan, Benjamin, Emma, Isabella, and James had the same rates as Alex and Elizabeth. The circuit and rate were given for Elizabeth, Benjamin, Olivia, Emma, Isabella, and James, and the students must find the amount of minutes. To tackle this challenge, students were reminded of the concept of 'the whole'. For Olivia's case, a half of the track at a rate of 18 minutes per full circuit of the track at rate of 9 minutes be circuit: $\frac{1}{2} \times 18 = 9$. Doubling and halving emerged for a discussion. Students might also discover that a runner's rate is faster because he or she ran for fewer minutes.

Conjecture of Students' Thinking and Expectation

In solving the problems, several big ideas and strategies are likely to emerge.

- Students might discover that to obtain the rate, they must divide (by fractions) and multiply (by the multiplicative inverse of fraction).
- Some students might recognize the numerical links and complete the chart using proportional reasoning. For example, the students can conclude that while Alex ran twice as long as Elizabeth and covered twice as much distance, his rate is the same as Elizabeth's: 30 minutes per circuit.
- Partial quotients might also be used, for instance for the case of Isabella, she did $\frac{3}{4}$ of the circuit in 15 minutes. Then, $\frac{3}{4}$ will be divided into 3 pieces and each piece was 5 minutes. So, the whole is 20 which we add 15 and 5, because that is $\frac{3}{4}+\frac{1}{4}$.
- Some students might use the multiplication and division relationship; for example, in the case of Elizabeth, the students might find the minutes first, as 2×30 equals 60. Alex covered the same distance in half the time. Then they considered $4 \times ? = 120$, and the answer will be 30, as well.
- Some students might utilize the double number line model, while others would use the chart as a ratio table, with arrows indicating rate relationships.

<u>Activity 4: Math Congress – The Marathon Training Results</u>

Learning Goal

Students share their ideas in solving problem in Activity 3 especially about the patterns on the chart and ideas related to multiplication and division with fractions.

Materials

Students' posters from Activity 3 and markers.

Description of the Activity

Before having a discussion with the researcher, the students were given the chance to discuss their strategies among others. This preliminary discussion aimed at allowing the students to consider several strategies they used to complete the training record chart. The questions that were asked to the students during their internal discussion, namely:

- Now that the training chart is complete, what connections do you observe and why do you think they formed?
- Which issues can be solved by dividing and which by multiplying?

After sufficient discussion time, the Math Congress was held. In this Math Congress, it was discussed:

- What is the whole?
- Can proportional reasoning be used to determine equal rates? Is it possible to solve the problem by multiplying the dividend (minutes) and divisor (circuits) by the same number?
- How do we know when to multiply and when to divide while working with fractions? What is the relationship between these two operations?

Proportional reasoning is the heart of this problem, which maintains equivalence while the ratio is kept constant. The students were expected to notice that Isabella ran $\frac{3}{4}$ of the track in 15 minutes and Emma ran $\frac{1}{4}$ of the track in 5 minutes, and the dividend and divisor have both tripled. If the students struggled with Olivia's and Emma's cases as they have different rates, they were expected to draw the track and mark out the fractions on the circuit.

After discussing the strategies used by the students to solve problems in Activity 3, students solved the string problems – Minilesson 2: Fractions as Operators (Figure 3.3). This Minilesson 2 encouraged students' flexibility of solving problem related to multiplication of fractions by a natural number. Beside solving the problems, the students were also invited to create their own contextual problem. Students might also be stimulated to use double number line model to represent their strategies.

String of related problem
$\frac{1}{2} \times 44$
$\frac{1}{4} \times 44$
$\frac{1}{8} \times 44$
$\frac{5}{8} \times 44$
$\frac{3}{8} \times 44$
$\frac{7}{12} \times 28$

Figure 3.7. Minilesson 2: Fractions as Operators

Activity 5: Exploring Playgrounds and Blacktop Areas

Learning Goal

Students solve the problem related to multiplication of fractions.

Materials

Worksheet 5, one sheet per student.

Description of the Activity

Students tried to solve problem given in the worksheet 5. The problem is as follows.

> There are two lots that will be changed into small parks in Cankaya area. Two of them are measures 50 meters by 100 meters. One lot, the residents agreed that $\frac{3}{4}$ of the lot will be devoted to a playground for children and then $\frac{2}{5}$ of that playground will be covered by blacktop, so children can play basketball. Another lot, the residents decided to use $\frac{2}{5}$ of the lot for a playground then $\frac{3}{4}$ of that will be blacktop.
> Is one lot getting more blacktop area than the other?

The problem above had been carefully designed in which the array model is expected to emerge to guide the student in solving multiplication of fractions problem. The numbers in the problem were also carefully chosen to develop the commutative property: $\frac{2}{5} \times \frac{3}{4} = \frac{3}{4} \times \frac{2}{5}$.

Through this activity, it was expected that the students would develop the array model to represent the relationship of playground to lot, blacktop to playground, and blacktopped-playground to lot.

Conjecture of Students' Thinking and Expectation

It was expected that students would discover that the blacktop areas are equivalent. Besides, it was also expected that the idea of equivalence versus congruence was discussed. Although the areas are equal, they may not be congruent depending on how students draw them. The strategies and challenges listed below could appear in students' solutions.

- Students cut both lots fourths vertically and fifths horizontally and then shaded or color the parts in which indicate the blacktopped area, $\frac{3}{4}$ of $\frac{2}{5}$ or $\frac{2}{5}$ of $\frac{3}{4}$. With this strategy, the ratio of the array of the blacktopped area (3 × 2) to the array of the lot (4 × 5). So, the blacktopped area of two lots will congruent as shown in Figure 3.8 below.

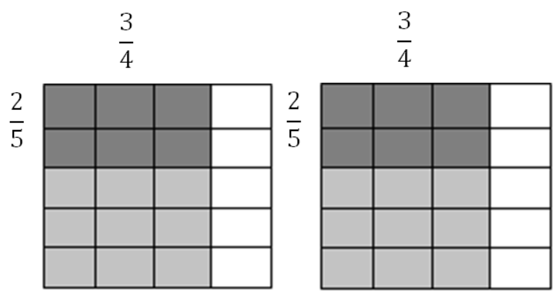


Figure 3.8. Blacktop areas of two lots (strategy 1)

- Students cut one lot into fourth horizontally then shaded $\frac{3}{4}$ indicating the playground, then marking fifths of that area vertically and shade $\frac{2}{5}$ of it the show the blacktop area. Similar way for the other lot, students would first cut the lot into fifths horizontally then shade $\frac{2}{5}$ indicating the playground, then marking fourths of that area vertically and shade $\frac{3}{4}$ of it the show the blacktop area (Figure 3.9). The thing that might challenge students through this strategy is that the areas are equivalent but non-congruent and students need to find a way to compare them.

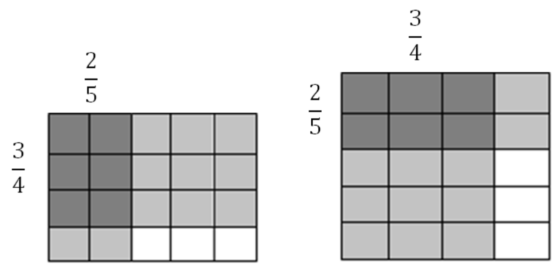


Figure 3.9. Blacktop areas of two lots (strategy 2)

- Students perhaps cut the fifths or fourths vertically (or horizontally). Students would find overlapping part but they might struggle to determine the fractional part.
- Students might use the dimensions of the lots which is 50 meters × 100 meters. This strategy may give result 37.5 meters × 40 meters, and 75 meters × 20 meters. Using this strategy, the students would find that the areas are equivalent but not congruent.

Activity 6: Math Congress – Exploring Playgrounds and Blacktop Areas

Learning Goal

Students share their ideas in solving problem in Activity 5.

Materials

Students' works from Activity 5 and Worksheet 6 (Minilesson 3).

Description of the Activity

Before having a discussion with the researcher, the students were given the chance to discuss their strategies among others. This preliminary discussion aimed at allowing the students to consider several strategies they used to solve problem in Activity 5.

After students had ample time to investigate and to solve the problem, the following ideas were discussed:

- The meaning of the term "of" as an indication of fraction multiplication.
- Multiplication's commutative property, which also applies to fractions.
- The array model's representation of the blacktopped area in relation to the total area of the lot.

After discussing the strategies and ideas underlying problems in Activity 5, students solved the string problems – Minilesson 3 (Figure 3.10) which consisted of a string of related problems of multiplication of fractions. This Minilesson 3 allowed students to solve the problem using the array model.

String of related problems:

$$\frac{1}{3}\times\frac{1}{5}$$

$$\frac{1}{3}\times\frac{3}{5}$$

$$\frac{2}{3}\times\frac{3}{5}$$

$$\frac{1}{6}\times\frac{3}{5}$$

$$\frac{5}{6}\times\frac{3}{5}$$

$$\frac{5}{6}\times\frac{4}{5}$$

$$\frac{7}{8}\times\frac{3}{4}$$

Figure 3.10. Minilesson 3: Multiplication of Fractions

<u>Activity 7: Comparing the Cost of Blacktopping</u>

Learning Goals

- Students use array model to solve the multiplication of fractions problem.
- Students extend their investigation with the commutative property of multiplication of fractions to percentages and decimals.

Materials

Worksheet 7 – one sheet per student.

Description of the Activity

This activity was the continuation of Activity 5 in which the students were asked to find the cost of blacktopping areas.

> Now the residents are considering the cost of the blacktopping in the playground area for both two lots. The cost of blacktopping in the first lot is $9 per square meters but the contractor will give offers to do it at 80% of that price, because it is a community project. In another lot, the contractor charges $8 per square meters but the contractor will give offers to do it at 90% of that price. Now the question is, will the blacktopping cost more in one of the parks than in the other?

Through this problem, students investigated strategies to multiply with equivalent forms of fractions – percentages and decimals – and extended their work with rational numbers. In Activity 8, students attempted to answer the problem and write down their strategies to be discussed in the Math Congress.

Conjecture of Students' Thinking and Expectation

As students go on to the problem of comparing the cost of blacktopping, several strategies were emerge:

- Before comparing the cost, the students calculated the area of blacktopping for each lot. The calculation depends on their drawing in Activity 6, whether they will calculate $\frac{3}{4}$ of 50 and $\frac{2}{5}$ of 100; or $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100. In calculating $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100, students will probably not have difficulty than calculating $\frac{3}{4}$ of 50. In calculating $\frac{3}{4}$ of 50, students might start with what they know, for instance by first find $\frac{1}{2}$ of 50 (=25) and $\frac{1}{2}$ of 25 (=12.5) which is same as $\frac{1}{4}$ of 50, and then they can calculate $\frac{3}{4}$ of 50 as 37.5 meters. Students might find the commutative properties that underlies the relationship of both blacktopping in two lots:

  37.5 meters × 40 yards = 75 meters × 1,500 yards = 1,500 square meters

  After students found the blacktopping areas of two lots, students might multiply the area with the price per meters and then subtract out the discount. For instance, for the first blacktopping area with \$9 per square meters and offers 80% of the price, students might multiply \$9 by 1,500 to determine the full price of blacktopping which is \$13,500; then to find 20% of \$13,500, students would use landmark of fraction in which 20% equal to $\frac{1}{5}$ and then calculated $\frac{1}{5}$ of \$13,500 (the discount, by dividing by 5, \$2,700); and subtracted that discount to get the final price of \$10,800. Similarly, to find the price of blacktopping in other lot uses \$9 per square meters and $\frac{1}{10}$ to calculate the discount.
- Some students directly included the discount in the calculation and used decimal or fraction forms to show the percentages (i.e., 80% = 0.8 = $\frac{8}{10}$; 90% = 0.9 = $\frac{9}{10}$). Then, calculated $0.8 \times \$9 \times 1{,}500$ square meters for the cost of first blacktopping area and calculated $0.9 \times \$8 \times 1{,}500\ \mathrm{m}^2$ for the cost of second blacktopping area. To find the calculation, the students might decompose the percentages or fractions through associative property:

$$\left(8 \times \frac{1}{10}\right) \times 9 \times 1{,}500 = 8 \times \left(\frac{1}{10} \times 9\right) \times 1{,}500$$

- Other students might think that there is no need to include the area as they found it is equivalent. They might just use $0.8 \times \$9$ and $0.9 \times \$8$. In this case, they might be challenged about why 80% of 9 = 90% of 8. It was expected that the students would find the associative property underlies the equivalence:

$$8 \times \left(10 \times \frac{1}{100}\right) \times 9 = 9 \times \left(10 \times \frac{1}{100}\right) \times 8$$

<u>Activity 8: Math Congress – Comparing the Cost of Blacktopping</u>

Learning Goals

- Students use interchanging numerators (or denominators) to derive the product of two fractions.
- Students share their ideas in solving problem in Activity 8 especially related to the commutative property and associative property of multiplication with fractions, percentages and decimals.

Materials

Students' works from Activity 7.

Description of the Activity

The activity was started by giving the students the Minilesson 4 (Figure 3.11) which consists of a string of related problems of interchanging numerators to multiply with fractions. This Minilesson 4 allowed students to make the problem 'friendlier', for instance $\frac{2}{3} \times \frac{3}{5} = \frac{3}{3} \times \frac{2}{5}$. This was string is designed to develop that strategy.

String of related problems:

$\frac{1}{3} \times \frac{1}{5}$

$\frac{2}{3} \times \frac{1}{5}$

$\frac{3}{5} \times \frac{2}{3}$

$\frac{2}{5} \times \frac{3}{3}$

$\frac{1}{5} \times \frac{1}{7}$

$\frac{2}{5} \times \frac{5}{7}$

$\frac{5}{5} \times \frac{2}{7}$

$\frac{3}{5} \times \frac{2}{3}$

Figure 3.11. Minilesson 4: Interchanging Numerators

In the Math Congress, students discussed several strategies that emerge in solving problem in Activity 8. The variety of strategies that were brought to a discussion, including:

- A 20% discount means that the price is now 80% of the original price.
- Using landmark or 'friendly' fractions can be useful strategy, for instance $10\% = \frac{1}{10}, 20\% = \frac{1}{5}, 25\% = \frac{1}{4}$ and $50\% = \frac{1}{2}$
- In determining equivalent fractions, percentages and decimals, rations must be kept constant, such as $80\% = 0.80 = 0.8 = \frac{8}{10} = \frac{4}{5}$
- Commutative and associative properties hold for multiplication of decimals and percentages, as well as for multiplication of fractions.

Regarding the cost of blacktopping areas, to encourage the students to generalize, two different representations could be helpful.

- Students used arrays to determine the equivalence. This array model provided opportunity for students to revisit the commutative property.

  $80\% \times 9 = 90\% \times 8$ because $\frac{72}{90} = \frac{72}{90}$ (Figure 3.12).

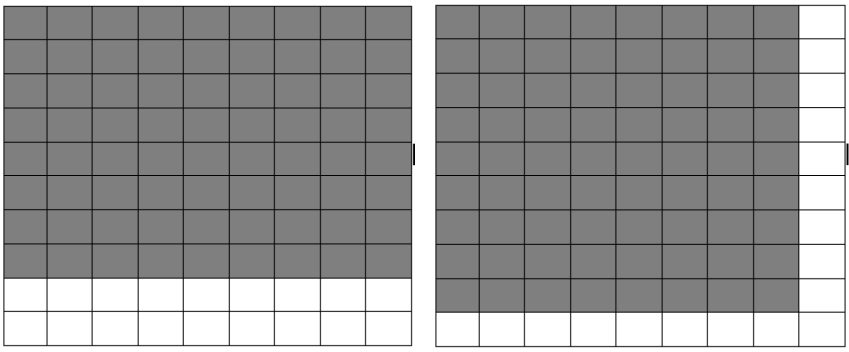


Figure 3.12. Array model representing $80\% \times 9 = 90\% \times 8$

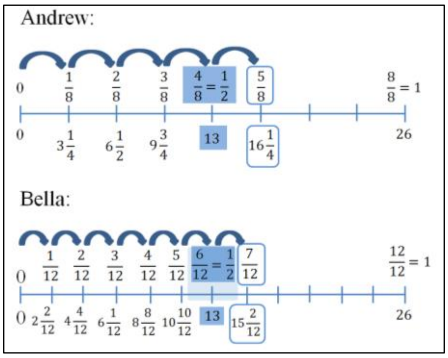
The numeric representation such as landmark percentages and the area of the blacktop to compute and compare the costs. For instance, students might argue that $80\% \times 9 = 90\% \times 8$ because $80\% \times 9 = \frac{8}{10} \times 9 = \left(8 \times \frac{1}{10}\right) \times 9 = 8 \times \left(\frac{1}{10} \times 9\right)$. This representation would provide students the opportunity to explore the associative strategy.

Table 3.3. Preliminary design of HLT for learning multiplication of fractions (Hellman & Fosnot, 2007; Holt et al., 2003)

Activities • **Running for Fun** • **Math Congress** • **Minilesson 1: Fractions as Operators**	
Learning Goals and Processes	• Students will develop several big ideas related to multiplication with fractions. • Students will share their ideas in solving problem especially in decomposing numbers and using partial products. • Students will revisit their strategies discussed in the math congress and use the idea of decomposing

	numbers and partial products to solve the string of Minilesson 1.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • To maintain equivalence, the ratio of the related numbers must be kept constant • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Model for Fraction	(Double) number line
Students' Possible Strategies	

- To find $\frac{5}{8}$ of 26 or $\frac{5}{8} \times 26$, students will try to find $\frac{1}{8}$ of 26 ($= 3\frac{1}{4}$) first, and then multiply $3\frac{1}{4}$ by 5 or using the idea of repeated addition in which $3\frac{1}{4}$ added five times. Students will use similar idea to find $\frac{7}{12}$ of 26 or $\frac{7}{12} \times 26$.
- Students may use the double number line to show repeated addition as shown below.



- By using distributive strategy, students may decompose $\frac{5}{8}$ into $\frac{4}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{8}$ and then multiplying it with 26, as explained below.
  $\frac{5}{8} \times 26 = \left(\frac{1}{2} + \frac{1}{8}\right) \times 26 = \left(\frac{1}{2} \times 26\right) + \left(\frac{1}{8} \times 26\right) = 13 + 3\frac{2}{8} = 13 + 3\frac{1}{4} = 16\frac{1}{4}$
  The distributive strategy as above will also be used to calculate $\frac{7}{12} \times 26$.
- Fractions which are difficult to work with can be decomposed into unit fractions.
- Equivalent of fractions, for instance: $\frac{1}{2} = \frac{4}{8} = \frac{6}{12}$.

<table>
<tr><td colspan="2">

- Equivalent fractions can be used to build multiplication strategies for fractions. For instance: $\frac{5}{8} = 5 \times \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$.
- The whole matters. Students cannot always compare the numerators or denominators while determining whether $\frac{7}{12}$ is more than $\frac{5}{8}$.
- For fractions, the distributive property holds for multiplication over addition. The partial product of $\frac{1}{2} \times 26$ and $\frac{1}{8} \times 26$ can be used to find the product of the whole, $\frac{5}{8} \times 26$.
- Through proportional reasoning, the students may determine that if $\frac{1}{8} \times 26 = 3\frac{1}{4}$, then $\frac{5}{8}$ of 26 = $5 \times 3\frac{1}{4} = 16\frac{1}{4}$.

</td></tr>
<tr><td colspan="2">

Activities

- **Training for Next Year's Marathon**
- **Math Congress**
- **Minilesson 2: Fractions as Operators**

</td></tr>
<tr><td>Learning Goals and Processes</td><td>

- Students will use landmark fractions and partial products when multiplying a fraction by a natural number.
- Students will explore multiplication and division (a natural number by a fraction) and the relationship between the operations.
- Students will share their ideas in solving problem in Activity 3 especially about the patterns on the chart and ideas related to multiplication and division with fractions.

</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>

- To maintain equivalence, the ratio of the related numbers must be kept constant
- The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers

</td></tr>
<tr><td>Model for Fraction</td><td>Ratio Table and Double Number Line</td></tr>
<tr><td colspan="2">Students' Possible Strategies</td></tr>
<tr><td colspan="2">

- Students may discover that to obtain the rate, they must divide (by fractions) and multiply (by the multiplicative inverse of fraction).

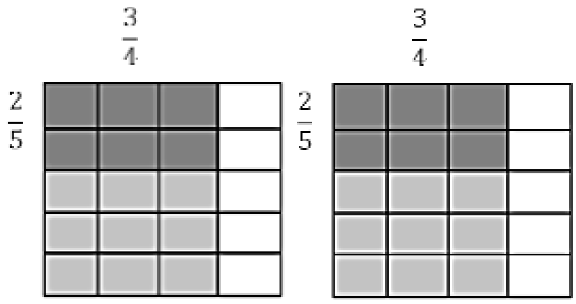
</td></tr>
</table>

• Some students may recognize the numerical links and complete the chart using proportional reasoning. For example, the students can conclude that while Alex ran twice as long as Elizabeth and covered twice as much distance, his rate is the same as Elizabeth's: 30 minutes per circuit. • Partial quotients may also be used, for instance for the case of Isabella, she did $\frac{3}{4}$ of the circuit in 15 minutes. • Then, $\frac{3}{4}$ will be divided into 3 pieces and each piece was 5 minutes. So, the whole is 20 which we add 15 and 5, because that is $\frac{3}{4}+\frac{1}{4}$. • Some students may use the multiplication and division relationship; for example, in the case of Elizabeth, the students may find the minutes first, as 2×30 equals 60. Alex covered the same distance in half the time. Then they will consider $4 \times ? = 120$, and the answer will be 30, as well. • Some students may utilize the double number line model, while others will use the chart as a ratio table, with arrows indicating rate relationships. • The students are expected to notice that Isabella ran $\frac{3}{4}$ of the track in 15 minutes and Emma ran $\frac{1}{4}$ of the track in 5 minutes, and the dividend and divisor have both tripled. • If the students struggle with Olivia and Emma's cases as their rates are differ, they are expected to draw the track and mark out the fractions on the circuit.
Activities • **Exploring Playgrounds and Blacktop Areas** • **Math Congress** • **Minilesson 3: Multiplication of Fractions**

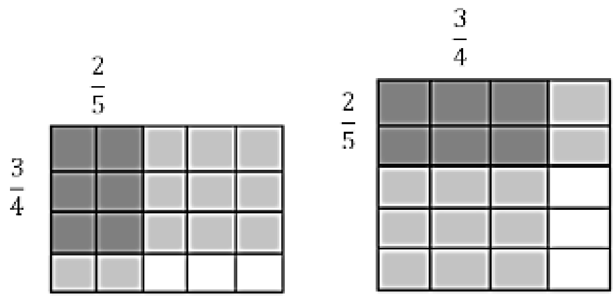
Learning Goals and Processes	• Students will solve the problem related to multiplication of fractions. • Students will share their ideas in solving problem in Activity 5.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Model for Fraction	Array/Area Model

Students' Possible Strategies

- Blacktop areas are equivalent.
- Students will cut both lots fourths vertically and fifths horizontally and then shade or color the parts in which indicate the blacktopped area, $\frac{3}{4}$ of $\frac{2}{5}$ or $\frac{2}{5}$ of $\frac{3}{4}$. With this strategy, the ratio of the array of the blacktopped area (3 × 2) to the array of the lot (4 × 5). So, the blacktopped area of two lots will congruent.



- Students will cut one lot into fourth horizontally then shade $\frac{3}{4}$ indicating the playground, then marking fifths of that area vertically and shade $\frac{2}{5}$ of it the show the blacktop area. Similar way for the other lot, students will first cut the lot into fifths horizontally then shade $\frac{2}{5}$ indicating the playground, then marking fourths of that area vertically and shade $\frac{3}{4}$ of it the show the blacktop area.



- Students perhaps cut the fifths or fourths vertically (or horizontally).
- Students may use the dimensions of the lots, which is 50 meters × 100 meters. This strategy may give result 37.5 meters × 40 meters, and 75 meters × 20 meters. Using this strategy, the students will find that the areas are equivalent but not congruent.

<table>
<tr><td colspan="2">Activities
Comparing the Cost of Blacktopping
Math Congress
Minilesson 4: Interchanging Numerators</td></tr>
<tr><td>Learning Goals and Processes</td><td>Students will use array model to solve the multiplication of fractions problem.
Students will extend their investigation with the commutative property of multiplication of fractions to percentages and decimals.
Students will use interchanging numerators (or denominators) to derive the product of two fractions.
Students will share their ideas in solving problem in Activity 8 especially related to the commutative property and associative property of multiplication with fractions, percentages and decimals.</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>Fractions represent a relation
The whole matters
The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers</td></tr>
<tr><td>Model for Fraction</td><td>Array/Area Model</td></tr>
<tr><td colspan="2">Students' Possible Strategies</td></tr>
<tr><td colspan="2">Before comparing the cost, the students will calculate the area of blacktopping for each lot.
The calculation depends on their drawing from previous activity, whether they will calculate $\frac{3}{4}$ of 50 and $\frac{2}{5}$ of 100; or $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100. In calculating $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100, students will probably not have difficulty than calculating $\frac{3}{4}$ of 50. In calculating $\frac{3}{4}$ of 50, students might start with what they know, for instance by first find $\frac{1}{2}$ of 50 (=25) and $\frac{1}{2}$ of 25 (=12.5) which is same as $\frac{1}{4}$ of 50, and then they can calculate $\frac{3}{4}$ of 50 as 37.5 meters.
Students might find the commutative properties that underlies the relationship of both blacktopping in two lots:
37.5 meters × 40 yards = 75 meters × 1,500 yards = 1,500 square meters</td></tr>
</table>

- After students find the blacktopping areas of two lots, students may multiply the area with the price per meters and then subtract out the discount. For instance, for the first blacktopping area with \$9 per square meters and offers 80% of the price, students might multiply \$9 by 1,500 to determine the full price of blacktopping which is \$13,500; then to find 20% of \$13,500, students will use landmark of fraction in which 20% equal to $\frac{1}{5}$ and then calculate $\frac{1}{5}$ of \$13,500 (the discount, by dividing by 5, \$2,700); and subtract that discount to get the final price of \$10,800. Similarly, to find the price of blacktopping in other lot uses \$9 per square meters and $\frac{1}{10}$ to calculate the discount.
- Some students will directly include the discount in the calculation and use decimal or fraction forms to show the percentages (i.e., 80% = 0.8 = $\frac{8}{10}$; 90% = 0.9 = $\frac{9}{10}$). Then, calculate $0.8 \times \$9 \times 1{,}500$ square meters for the cost of first blacktopping area and calculate $0.9 \times \$8 \times 1{,}500$ square meters for the cost of second blacktopping area. To find the calculation, the students might decompose the percentages or fractions through associative property:

$$\left(8 \times \frac{1}{10}\right) \times 9 \times 1{,}500 = 8 \times \left(\frac{1}{10} \times 9\right) \times 1{,}500$$

- Other students may thing that there is no need to include the area as they found it is equivalent. They may just use 0.8 × \$9 and 0.9 × \$8. In this case, they might be challenged about why 80% of 9 = 90% of 8. It is expected that the students will find the associative property underlies the equivalence:

$$8 \times \left(10 \times \frac{1}{100}\right) \times 9 = 9 \times \left(10 \times \frac{1}{100}\right) \times 8$$

- A 20% discount means that the price is now 80% of the original price.
- Using landmark or 'friendly' fractions can be useful strategy, for instance $10\% = \frac{1}{10}, 20\% = \frac{1}{5}, 25\% = \frac{1}{4}$ and $50\% = \frac{1}{2}$
- In determining equivalent fractions, percentages and decimals, rations must be kept constant, such as $80\% = 0.80 = 0.8 = \frac{8}{10} = \frac{4}{5}$
- Commutative and associative properties hold for multiplication of decimals and percentages, as well as for multiplication of fractions.
- Students may use arrays to determine the equivalence. This array model will provide opportunity for students to revisit the commutative property.

$80\% \times 9 = 90\% \times 8$ because $\frac{72}{90} = \frac{72}{90}$

- The numeric representation such as landmark percentages and the area of the blacktop to compute and compare the costs. For instance, students may argue that $80\% \times 9 = 90\% \times 8$ because $80\% \times 9 = \frac{8}{10} \times 9 = \left(8 \times \frac{1}{10}\right) \times 9 = 8 \times \left(\frac{1}{10} \times 9\right)$. This representation will provide students the opportunity to explore the associative strategy.

3.4. Research Subject and Researcher's Role

This study featured five students from an international school in Ankara, Türkiye, where the instructional activities of learning multiplication of fractions designed based on RME were tested. Two students, denoted as P1 and P2, participated in the pilot experiment, while three students, identified as S1, S2, and S3, were involved in the teaching experiment.

The choice of the specific school for this study was deliberate. The school was selected due to its adherence to an American-based curriculum that incorporated a constructivist approach to teaching and learning. This curriculum was aligned with the principles at the core of the RME approach, which aimed to foster students' active engagement in the development of mathematical concepts and skills. The school's commitment to this pedagogical framework created an ideal environment to investigate the effectiveness of the instructional activities designed based on RME approach.

Moreover, the school conducted all its instruction in English, which facilitated the research process. The researcher, who served as the instructor throughout the study, was able to seamlessly communicate and interact with the students. This language consistency eliminated potential language barriers and ensured the smooth execution of the research activities.

The selection of an international school with a constructivist curriculum and English as the language of instruction provided a suitable context for investigating the impact of the RME approach on students' mathematical thinking and learning outcomes.

The pilot experiment in this study was specifically designed to assess the feasibility of the activities included in the initial HLT. Its primary objective was to identify potential challenges faced by students and gather their ideas in order to enhance the initial conjecture of students' thinking. The pilot experiment involved two students who were not part of the actual teaching experiment, ensuring a separate test run for the HLT.

The selection of these two students was based on their willingness to participate in the research program and the consent provided by their parents. Prior to their involvement, both the students and their parents were fully informed about the purpose and nature of the study, including the fact that their information would be utilized in a doctoral dissertation. This transparent approach ensured that the participants and their parents were aware of the implications of their involvement and gave informed consent.

Furthermore, the preliminary information obtained from the students' regular teacher indicated that they were high and middle-achieving students, consistently performing well in mathematics in their regular classes. This information suggested that the selected students had a strong foundation in mathematical proficiency, making them suitable candidates for exploring the potential effects of the HLT on students with varying levels of achievement.

The engagement of these specific students in the pilot experiment allowed for valuable insights into the initial efficacy and potential areas of improvement for the HLT, paving the way for further refinement and adjustments in the subsequent teaching experiment.

The teaching experiment in this study involved three students who were selected based on their willingness to participate in the research program and the consent provided by their parents, ensuring a voluntary and informed involvement. Similarly to the pilot experiment, the goal was to investigate the effectiveness of the HLT in learning multiplication of fractions.

The students selected for the teaching experiment were deliberately chosen to represent a spectrum of achievement levels. According to information provided by their regular teacher, the three students included high, middle, and low-achieving students. This intentional variation in achievement levels was in line with the researcher's objective of exploring whether the designed HLT could be effective for all students, regardless of their initial proficiency in mathematics. By including students across the achievement spectrum, the study aimed to determine the potential impact of the designed HLT on students with different levels of mathematical ability.

This design allowed for a comprehensive examination of the effectiveness and adaptability of the HLT, considering the range of students' prior mathematical knowledge and skills. By including students with diverse achievement levels, the researcher sought to develop a nuanced understanding of how the HLT activities could support and enhance the multiplication of fractions learning process for a variety of students.

During the implementation of pilot and teaching experiments, which were conducted after school, the researcher served as the instructor. As the instructor, the reasearcher took charge of leading the entire research process, which involved the implementation of instructional activities and the collection and analysis of data. This active involvement allowed the researcher to have direct control over the research procedures and ensure adherence to the research design.

However, it is important to note that the real classroom teacher did not actively participate in the research process. The researcher received permission from the school management, which restricted the involvement of the classroom teacher.

While the absence of the classroom teacher's involvement may limit the generalizability of the findings to a broader teaching context, it allows the researcher to have a clearer distinction between the regular classroom practices and the research activities being implemented. This separation reduces the potential influence of the teacher’s existing pedagogical beliefs or instructional approaches on the research outcomes.

By having control over the implementation of the activities and data collection process, the researcher can maintain consistency and accurately document the effects of the interventions being tested. The researcher’s direct involvement also helps in ensuring the reliability and validity of the research findings.

To ensure the smooth execution of the research, two assistants were also present to support the researcher. These assistants played a significant role in recording videos of the classroom interactions and capturing critical moments during the implementation of the sequence of activities. Their assistance in documenting these key moments was instrumental in enriching the data collection process and ensuring the accuracy and reliability of the study's findings.

The researcher, along with the two assistants, worked collaboratively as a team throughout the study. Their combined efforts allowed for a comprehensive and well-documented research process, ensuring that no significant events or interactions during the implementation of the activities were missed.

This collaborative approach involving the researcher and the assistants contributed to the overall quality and rigor of the study and provided a robust foundation for the data analysis and interpretation. The arrangement of having multiple members involved in the research process not only facilitated the smooth running of the study but also allowed for different perspectives and insights to be considered during data analysis and interpretation.

3.5. Research Timeline

Table 3.4 below summarizes the timeline of the study.

Table 3.4. Research timeline

	Date	Description
Preliminary Design		
Literature review and design HLT.	August 2014 – January 2015	
Pilot Experiment (Phase 1)		
Observing grade 5 students (the class where students will be involved in the pilot experiment)	2 February 2015	From the observation, we selected two students to implement the initial HLT in small group.
Trying out the learning activities	9 – 18 February 2015	Tried out the questions/problems in the worksheets.
Teaching Experiment (Phase 2)		
Activity 1: Running for fun	23 February 2015	Investigated the presence of big ideas and strategies of multiplication of fraction by natural number. It was expected that double number line would appear through solving the problem of Running for Fun.
Activity 2: Math Congress – Running for fun and Minilesson	25 February 2015	Shared and discussed ideas and strategies resulted from solving problem in Activity 1. Solved problems in the Minilesson “Fractions as Operator”.
Activity 3: Training for next year’s marathon	27 February 2015	Developed the landmark of fractions and partial products. It was expected that through this context, the ratio table model would appear.

Activity 4: Math Congress – The marathon training results, and Minilesson	2 March 2015	Shared ideas and strategies resulted from solving problem in Activity 3. Solved problems in the Minilesson “Fractions as Operator”.
Activity 5: Exploring playgrounds and blacktop areas	4 March 2015	Investigated the presence of big ideas and strategies of multiplication of fraction by fraction. It was expected that through this context, the array model would appear.
Activity 6: Math Congress – The blacktop areas, and Minilesson.	9 March 2015	Shared ideas and strategies resulted from solving problem in Activity 4. Solved problems in the Minilesson “Multiplication of Fractions”.
Activity 7: Comparing the cost of blacktopping	13 March 2015	Investigated the use of array model for multiplication of fractions.
Activity 8: Discussion – Comparing the cost of blacktopping, and Minilesson	16 March 2015	Shared ideas and strategies resulted from solving problem in Activity 7. Solved problems in the Minilesson “Interchanging Numerators”.

3.6. Data Sources and Data Collection

For the purpose of this study, various sources of data were utilized, including students' works, researcher field notes, and video recordings of learning moments. These data collection methods were employed to comprehensively capture and analyze the students' experiences and learning outcomes during the implementation of the instructional activities designed based on RME approach.

Video recordings played a particularly significant role in documenting the students' learning activities. Each student's learning activity was captured by three video cameras strategically placed in the classroom. One camera was positioned in the front of the classroom, providing a comprehensive view of the overall classroom dynamics and interactions. Another camera was positioned in the back of the classroom, capturing a different perspective and facilitating a thorough observation of the students' engagement and participation.

Additionally, the researcher utilized a third video camera to document crucial moments that occurred when students collaborated and interacted with their peers. This camera was specifically aimed at capturing the unique social dynamics and group interactions during the learning activities.

By utilizing multiple video cameras from different angles, this study aimed to ensure a comprehensive and detailed documentation of the students' learning process. The collection of video data allowed for a close analysis of the students' engagement, interactions, and problem-solving approaches, providing valuable insights into the effectiveness of the RME approach in learning multiplication of fractions.

The data obtained through video recordings, students' works, and field notes were essential in gaining a comprehensive understanding of the social and sociomathematical norms that emerged during the learning activities. These data sources provided valuable insights into the students' individual learning processes and the dynamics of their peer interactions.

To ensure a thorough analysis of the video recordings, each recording was transcribed, converting the spoken language and non-verbal cues into written form. Transcribing the videos allowed for a closer examination of the students' communication patterns, problem-solving strategies, and mathematical reasoning.

Additionally, at the end of each learning activity, the students' works, such as worksheets or problem-solving tasks, were collected. The analysis of these worksheets during the math congress sessions allowed the researcher to gain insights into how the students approached and solved mathematical problems. This examination also shed light on how the students applied their strategies during peer discussions, providing important information on their collaborative problem-solving skills.

To complement the video recordings and students' works, field notes were collected by the researcher. These field notes served as a reflective tool, allowing the researcher to record immediate thoughts, observations, and impressions during and after the learning activities. The collection of field notes enriched the data collection process and provided a contextual background for the analysis. The inclusion of multiple data sources, such as video recordings, students' works, and field notes, added depth and rigor to the study. It would also enriched data and provide a rich context for analysis (Creswell, 2013; Lofland et.al., 2005; Patton, 2002).

The data collection period for this study spanned approximately four months, beginning prior to the design of the instructional sequence on multiplication of fractions. The purpose of the initial data collection phase, termed the pilot experiment (phase 1), was to try out the designed activities with two students (P1 and P2) and gather insights into their strategies and struggles when solving the problems. This data would then be used to improve the initial HLT, including refining the conjecture of students' thinking.

Following the improvements made to the initial HLT approach, the teaching experiment phase (phase 2) was conducted, involving three students (S1, S2, and S3). In this phase, the same data collection methods as in the pilot experiment were employed. Students' works, researcher field notes, and video recordings were collected to comprehensively capture and analyze the students' experiences and learning outcomes.

Table 3.5 provided a detailed overview of the data collection process, describing the specific data sources utilized during the phases of the study.

Table 3.5. Data collection

Phases	Data Collection	Description
Preliminary Design	Classroom observation in grade 5	Investigating students' current knowledge about fractions and multiplication of fractions.
Pilot Experiment (Phase 1)	Trying out the learning activities to two grade 5 students (Video recordings, field notes, and students' works)	• Testing the initial HLT. • Investigating students' strategies in solving the problems. • Improving the initial HLT including the conjecture of students' thinking.
Teaching Experiment (Phase 2)	Trying out the learning activities to three grade 5 students (Video recordings, field notes, and students' works)	• Testing the improved HLT. • Investigating students' strategies in solving the problems and compare it with the conjecture of students' thinking.
	Discussions with students Video recording	Finding students' remarks about the whole teaching and learning process and special moment(s) which occurs in the process of learning activities.

3.7. Data Analysis

Having access to a large amount of data, specifically video recordings that totaled approximately 25 hours of classroom implementation from both the pilot and teaching experiments, necessitated a thorough and organized approach to data analysis. While the data had been continuously studied as part of the ongoing analysis process, the retrospective analysis aimed to gather information on the students' learning of multiplication of fractions and track their mathematical thinking and practices. To achieve these objectives, transcripts of the data from the classroom videos were compiled and served as the starting point for analysis.

Given the nature of the design experiment that formed the foundation for constructing theoretical frameworks (Cobb et al., 2003), the data were analyzed using the grounded theory approach known as constant comparative data analysis. This approach allows for the construction of concepts from the data, employing a method known as the constant comparative method (Taylor & Bogdan, 1998). This method involves coding and evaluating the data simultaneously, enabling the researcher to iteratively analyze and compare different instances of data to identify patterns, themes, and emergent theories.

In addition to the video recordings, other sources of data, such as observations, field notes, and documentation of students' works, were also utilized in the analysis process. These multiple data sources provided a comprehensive understanding of the students' learning experiences and facilitated triangulation, strengthening the validity and reliability of the findings.

The constant comparative method is a systematic approach that combines data collection, coding, and analysis with theoretical sampling to generate theory that is closely connected to the data and can be further tested (Conrad et al., 1993). The method was developed with the aim of producing theory that is integrated, grounded in the data, and expressed in a clear form that can be subject to further investigation. The constant comparative approach consists of four stages: comparing incidents applicable to each category, integrating categories and their properties, delimiting the theory, and finally, writing the theory (Glaser & Strauss, 1967).

Throughout the four stages of the constant comparative technique, the researcher continuously engaged in sorting through the collected data, performing analyses, and coding the material. The process of theoretical sampling, which involves purposefully selecting data to explore and refine emerging theories, was also a key aspect of this method (Glaser & Strauss, 1967). The advantage of employing this methodology is that it allowed for the development of a plausible theory based on raw data through ongoing comparisons and refinements.

Succeeding at the development of a grounded theory using the constant comparative method required significant time and attention from the researcher, who had to dedicate themselves to the procedures of data collection and analysis. This time-consuming endeavor was crucial for constructing a theory that was deeply rooted in the empirical evidence and accurately reflected the experiences and perspectives of the participants.

Rasmussen and Stephan (2008) developed a methodology aimed at investigating collective learning in the classroom setting. The objective of their approach was to capture and analyze the shared learning activities that take place in the classroom, including the facilitation of mathematics learning and whole-class discussions. The methodology proposed by Rasmussen and Stephan consists of three distinct phases that enable the documentation of participant thinking and the assessment of classroom discourse. This approach also involved the selection of mathematical procedures that were widely recognized as common knowledge.

The three phases of the methodology encompass a range of activities, purposes, and concepts, all of which contribute to the generation of various outcomes resulting from the application of different strategies and procedures. These variations in approach and content allow for the development of a diverse array of products, reflecting the concepts, strategies, and procedures utilized in the classroom. The progression of mathematical practices throughout these phases is illustrated in Table 3.5, providing a visual representation of the evolution of learning processes (Rasmussen & Stephan, 2008).

By employing this methodology, Rasmussen and Stephan sought to uncover and understand the mechanisms by which collective learning takes place in the classroom context. Through the systematic analysis of classroom activities and discourse, this approach allows researchers to gain valuable insights into the dynamics of shared learning and the development of mathematical practices.

Table 3.6. Phases in documenting collective (Rasmussen and Stephan, 2008, p. 83-84)

Phases of Research	Activity	Product
Phase One	• Transcribe every whole class discussion • Notate claims made by students or researcher • Identify data and conclusions • Compare argumentation schemes and come to agreement	Argumentation Log
Phase Two	• Use Argumentation Log as data • Identify taken-as-shared mathematical ideas	Mathematical Ideas Chart
Phase Three	• Use Mathematical Ideas Charts to identify common mathematical activities associated with taken-as-shared mathematical ideas	Classroom Mathematical Practices

In the initial phase of the methodology developed by Rasmussen and Stephan (2008), the videotape recordings of each instructional activities in this study were transcribed, creating transcripts that documented all class discussions. These transcripts served as the basis for analysis. The generation of an argumentation log marked the completion of this first phase.

To ensure the reliability and validity of the analysis process, multiple researchers working independently held meetings to extract and identify relevant information from the transcripts. In this process, the additional assistants who were not involved in the teaching sessions took on the role of witnesses and observers. Independently, they produced their own argumentation logs, which included claims and corresponding data for each claim. Subsequently, the assistants who served as producers of the argumentation logs engaged in discussions to compare and reconcile their respective analyses. They would either agree or disagree with each other's viewpoints until a consensus was reached. In cases where consensus was not initially achieved, further discussion and deliberation were conducted until agreement was attained. This iterative process of analysis and consensus-building further reinforced the overall analysis, enhancing the reliability of the findings.

The collaborative nature of the analysis process, involving multiple researchers and assistants, allowed for different perspectives and insights to be considered. By engaging in discussions and converging on an agreement regarding the argumentation logs and their components, the researchers ensured a robust and comprehensive analysis of the transcripts and the claims made within them.

This rigorous methodology employed by Rasmussen and Stephan (2008) for analyzing transcripts and generating argumentation logs allowed for a systematic and objective approach to capturing and understanding the classroom discourse and the claims made within it.

The resulting argumentation logs were used in the second phase of the study, which focuses on the identification of mathematical ideas that are assumed to be taken-as-shared. The aim of this phase was to extract evidence from the argumentation logs to determine which mathematical ideas were becoming accepted and shared among the participants, as demonstrated through data gathered across all teaching episodes and class discussions.

During this stage of analysis, the researcher employed a systematic approach to extract mathematical ideas from the argumentation logs. Rasmussen and Stephan (2008) proposed two criteria to determine when a mathematical idea could be considered as taken-as-shared. Firstly, it was deemed that when all participants reached a shared understanding of a particular mathematical idea, they would no longer challenge the argumentation surrounding it. This criterion indicated that the participants had accepted the idea as valid and were no longer engaged in questioning or challenging its significance.

Secondly, the researcher identified that a mathematical idea could be considered taken-as-shared when it became self-evident within the arguments and subsequently used as a reason or justification in subsequent arguments. This criterion highlighted the idea that if a mathematical concept consistently emerged as a supporting element in different arguments, it could be seen as a shared understanding and perspective among the participants.

By applying these criteria to analyze the argumentation logs, the researcher was able to identify the mathematical ideas that had transformed into taken-as-shared concepts within the classroom setting. Through this rigorous analysis process, they were able to provide insights into the development and dissemination of mathematical knowledge within the learning environment.

Rasmussen and Stephan (2008) proposed the use of a mathematical ideas chart as a tool for researchers to effectively identify and track the mathematical practices employed during the instructional sequence. This chart, consisting of three columns, was designed to document the status of mathematical ideas within each lesson.

The first column of the chart is dedicated to recording the mathematical ideas that are currently functioning as-if-shared. These ideas are the ones that have already been established and accepted by the participants as valid within the argumentation. The second column is reserved for the mathematical ideas that were discussed during the lesson and are being observed to see if they subsequently function as if they are shared.

This column is important for tracking the development of mathematical ideas and capturing any shifts in their status over time. Finally, the third column serves as a space for additional comments that may provide further insights or explanations related to the mathematical ideas under consideration (Rasmussen & Stephan, 2008).

To illustrate the application of this chart, Rasmussen and Stephan (2008) provided an example for the first activity of this research study, which is displayed in Table 3.6. By using this chart, researcher was able to systematically analyze and compare multiple lessons to facilitate the emergence of taken-as-shared ideas in a progressive manner. Moreover, this tool allowed for the identification of the transition of mathematical ideas from the "keep an-eye-on" category to the "taken-as-shared" category, along with capturing any additional comments that provided theoretical or practical implications.

By employing the mathematical ideas chart, researcher could effectively track and document the development and dissemination of mathematical ideas within the instructional setting. This enhanced the researcher's understanding of how mathematical practices were integrated into the classroom context and how certain ideas became accepted and shared among the participants.

Table 3.7. Mathematical ideas chart for the first activity, i.e., Running for Fun context

Mathematical ideas that function as-if-shared	Mathematical ideas to keep an-eye-on	Additional comments
• Identification of fractions positioned along the running route (the whole matters). • Identification of halfway point of the running route indicated by marker	• Connecting the running route to a number line as a model from measurement context. • Equivalent fractions when the fractions are alligned on the same line (to maintain equivalence, the ratio	• With the aid of the image of the running route, the position of fractions on the running route, the halfway point, and whether $\frac{7}{12}$ and $\frac{5}{8}$ are greater than the halfway point were examined.

and water bottle (fraction represent a relation). • Identification whether $\frac{7}{12}$ and $\frac{5}{8}$ are greater than the halfway point (comparing fractions). • The possibility of determining how far Bella and Andrew ran (multiplication of fractions with natural numbers utilizing several strategies, such as decomposing numbers and using partial products).	of related numbers must be kept constant). • Development of strategies for multiplication of fractions with natural numbers.	• The strategies used to find the distance of how far Bella and Andrew ran were investigated. The strategies could then be led to the concept of multiplication of fractions with natural numbers.

In the third phase of the research process, the focus shifted towards determining the taken-as-shared mathematical ideas within the instructional setting. This phase drew upon the framework proposed by Cobb and Yackel (1996) and further developed by Rasmussen and Stephan (2008). Taken-as-shared mathematical ideas refer to those ideas that have been collectively accepted and embraced by the participants as valid and relevant within the mathematical discourse.

To identify these taken-as-shared mathematical ideas, researcher engaged in a systematic process of identification, analysis, and categorization. This involved closely examining the contexts and situations in which the ideas emerged and became established as taken-as-shared. By studying these specific contexts and ideas, researcher was able to draw connections and categorize the mathematical ideas according to their underlying concepts and representations. Additionally, the researcher examined the mathematical activities in which the participants were actively involved, as these activities played a crucial role in the development and dissemination of the taken-as-shared ideas.

The outcome of this phase was the identification of the specific classroom mathematical practices that derived from the taken-as-shared ideas. These practices encompassed the ways in which the participants engaged with and applied the mathematical ideas within their learning processes. By analyzing the taken-as-shared ideas and linking them to the corresponding mathematical activities, researchers gained insight into the classroom-level practices that shaped mathematical thinking and reasoning.

The studies conducted by Cobb and Yackel (1996) and Rasmussen and Stephan (2008) underline the significance of identifying taken-as-shared mathematical ideas and their relationship to instructional practices. These studies highlight the role of contextual factors, such as the nature of mathematical discourse, collaborative problem-solving activities, and teacher facilitation, in the development and dissemination of mathematical ideas within the classroom.

3.8. Trustworthiness

The students' understanding of multiplication of fractions was analyzed through multiple instructional activities, and data were collected from various sources, including observations, field notes, documents, and students' works. Collecting data from multiple sources is crucial to ensure trustworthiness in the research. Although there is no universal best method to achieve trustworthiness in research, triangulation is often used to achieve a broader understanding of a topic, disclose different dimensions of areas of interest, and verify findings by putting methods in conversation with one another.

Triangulation is a multi-methodological perspective that aims to describe a specific situation from various points of view, combining both qualitative and quantitative methods. According to Munday (2009), triangulation presents a mixed-methods approach to create a comprehensive understanding of social contexts. It provides a way to check scientific results by combining research methodologies, theories, or data sources. Triangulation in research has been employed by many researchers in different domains.

For example, Carl (2009) used triangulation to investigate human translation processes by using combined research methodologies, theories, or data sources to gain a better understanding of the various stages of the translation process. Hansen (2003, 2009, 2010) employed a triangulation of process- and product-oriented research approaches, using questionnaires, interviews, translations, revisions, and retrospective processes to investigate the many aspects of the translation process. According to Saldanha and O'Brien (2013), methodological triangulation is recognized as the foundation of solid and high-quality research.

Triangulation is a research methodology that involves gathering evidence from multiple sources to enhance the reliability and validity of a study (Denzin, 1978; Flick, 2004). By utilizing different data sources, such as observations, field notes, documents, and students' works, researchers can corroborate and validate their findings, therefore strengthening the trustworthiness of their research design (Creswell, 2014).

In the context of the current study, triangulation was employed as a means of collecting data from diverse perspectives and sources. This approach is valuable for identifying different dimensions of the areas of interest and providing a more comprehensive understanding of the topic at hand. For example, observations allowed for a direct assessment of classroom activities, interactions, and behaviors, while field notes provided subjective insights and interpretations. Additionally, documents such as lesson plans and curriculum materials provided valuable contextual information, and students’ works offered tangible evidence of their learning outcomes. By integrating and analyzing data from these various sources, researchers could establish a more solid foundation for their conclusions and interpretations (Flick, 2004; Denzin, 1978).

Member-checking is another strategy employed in the research process to enhance trustworthiness and credibility (Creswell, 2009). This strategy involves returning the data, descriptions, themes, or interpretations derived from the analysis back to the individuals who participated in the study to validate the findings (Yardley, 2000). In the present study, follow-up interviews were conducted with the students who were involved in the research. By discussing the results, conclusions, and interpretations with the participants, the researchers provided an opportunity for clarification, correction, or validation of the data. This iterative process helps to ensure that interpretations are accurate and reflective of the participants' perspectives and experiences. Furthermore, member-checking encourages cooperation between researchers and participants, fostering a sense of trust and collaboration (Creswell, 2009; Yardley, 2000).

In addition to employing triangulation and member-checking, detailed descriptions were also utilized in this study to effectively convey the findings and provide crucial contextual information (Merriam & Tisdell, 2016). By including rich descriptions of the study setting, the researcher ensured that readers could better understand the context and conditions under which the research took place. This level of detail allowed readers to relate to the study and potentially draw connections with their own experiences, making the results more realistic and applicable (Merriam & Tisdell, 2016).

Detailed descriptions help to establish trustworthiness by providing transparency and clarity about the research process, the participants involved, and the methods utilized (Creswell, 2014). They allow readers to critically evaluate the study's findings and assess the transferability of the research to other contexts (Merriam & Tisdell, 2016). Moreover, the inclusion of detailed descriptions demonstrates the researcher's commitment to maintaining a rigorous approach, increasing the trustworthiness and credibility of the research (Creswell, 2014).

The use of triangulation in this study serves to enhance the robustness of the research design by using multiple data sources and methods to validate the findings (Denzin, 1978; Flick, 2004). Triangulation helps to address potential biases, increase the reliability of the results, and enhance the generalizability of the findings (Denzin, 1978; Flick, 2004). By integrating observations, field notes, documents, and student work, the researcher benefitted from different perspectives and data types, thus minimizing the risk of drawing conclusions based on a single source of evidence (Creswell, 2014).

Triangulation not only enhances the validity and reliability of the research results but also contributes to the overall credibility of the study. By systematically collecting and analyzing data from multiple sources, triangulation helps to create a more complete and comprehensive picture of the research topic (Flick, 2004). In the context of this study, triangulation allowed the researcher to verify the results obtained, identify different dimensions of the students' understanding of multiplication of fractions, and provide a more robust and nuanced understanding of the topic (Flick, 2004).

Therefore, the use of multiple data sources, detailed descriptions, triangulation, and member-checking in this study contributes to its trustworthiness, rigor, and validity. These strategies highlight the researcher’s methodological rigor, establishes the credibility and reliability of the research, and ensures that the findings can be relied upon for future use (Creswell, 2009; Denzin, 1978; Flick, 2004; Merriam & Tisdell, 2016).

3.9. Limitation of the Study

The present study had a few limitations that need to be acknowledged. The primary limitation of the study stemmed from the design of qualitative research and the method of design-based research that the study employed. Due to the qualitative nature of the study, it was challenging to generalize the findings to other contexts or populations beyond those of the study participants. This limitation made it difficult to determine the broader implications of the study's findings for teaching and learning of mathematical concepts beyond the scope of the current research.

The second limitation of the study was that its conclusions focused mainly on the participants' individual contributions to the overall learning of mathematical concepts. While the study explored the social dimensions of the emergent perspective, it did not adequately consider the participants' individual educational development. This limitation restricted the scope of the study, preventing it from offering a comprehensive understanding of how students' thinking developed throughout the research process.

The third limitation of the study was the narrow focus on specific contexts for teaching and learning multiplication of fractions, namely the use of number line, ratio table, and array models (Cai & Hwang, 2019). While these contexts provided valuable insights into students' understanding within those specific frameworks, they may not capture the full range of ways in which multiplication of fractions can be taught and learned.

By limiting the study to only four specific contexts, there is a risk of overlooking alternative contexts that could potentially yield different patterns of understanding or highlight unique challenges that students might face in different instructional settings (Sowder, 1992). For example, other contexts such as real-world applications (e.g., baking recipes, scaling ingredients), area models, or even visual representations like pie charts or bar graphs could provide additional perspectives on students' understanding of multiplication of fractions.

To overcome this limitation, future research should consider including a broader range of contexts to gain a more comprehensive understanding of how students perceive and conceptualize multiplication of fractions across various instructional approaches. By examining multiple contexts, researchers can uncover commonalities or differences in students' understanding, thereby enhancing the generalizability of the findings and their applicability to different educational contexts (Cai & Hwang, 2019).

Additionally, exploring a wider variety of contexts can also shed light on potential misconceptions or difficulties that may arise when teaching multiplication of fractions. This knowledge can inform instructional strategies and curriculum design, ensuring that students develop a robust understanding of this complex mathematical concept (Torbeyns et al., 2008).

It is important to acknowledge that selecting specific contexts for a study is a common practice in research and can provide valuable insights within those chosen contexts (Cai & Hwang, 2019). However, recognizing the limitations of such an approach is crucial for the overall advancement of knowledge in the field of mathematics education.

Another limitation of the study was related to the role of the researcher as an instructor. The researcher served as a facilitator and played an active role in guiding participants in the discovery of mathematical concepts. This role may have affected the process of the emergence of mathematical ideas. Therefore, it is challenging to determine whether the outcomes of the study were a result of the researcher's influence or the participants' natural learning processes.

Despite these limitations, this study provides valuable insights into the processes of students' learning and the potential benefits of design-based research in educational settings. The findings offer valuable suggestions for future research, including the need for larger-scale studies that incorporate mixed-method research designs to overcome the generalization limitations of qualitative research (Creswell & Plano Clark, 2018). Additionally, future studies should consider the potential influence of researcher interventions on the learning process and explore how they can be minimized to help provide a more accurate representation of students' natural learning processes (Guba & Lincoln, 1994).

Moreover, although the research was conducted in an international school in Ankara, Türkiye, which used an American-based curriculum, this study could contribute to the mathematics classroom in Turkish-based curriculum as well. In the Turkish curriculum, the multiplication of fractions is taught in grade 6 under the topic "M.6.1.5. Operations with Fractions" (T.C. Milli Eğitim Bakanlığı, 2018). The use of the number line model in this study is also in line with the Turkish curriculum's expectation for students to "M.6.1.5.1. Compares, sorts, and displays fractions on the number line". Additionally, the Turkish curriculum suggests using real-life situations to learn the multiplication of fractions with natural numbers (M.6.1.5.3) and the multiplication of two fractions (M.6.1.5.4).

By aligning with the objectives and recommendations of the Turkish curriculum, this study's findings can serve as valuable insights for Turkish mathematics educators and curriculum designers to enhance the teaching and learning of multiplication of fractions. Implementing instructional strategies that incorporate the number line model and real-life situations can help foster a deeper understanding among students and support their ability to apply mathematical concepts in practical contexts.

3.10. Summary

The research study employed a design research methodology to investigate the understanding of multiplication of fractions among students. This approach is commonly used in educational research to develop and evaluate instructional interventions in real educational settings (Cobb et al., 2003). In this study, a Hypothetical Learning Trajectory (HLT) was developed and validated based on the results of a pilot study. The purpose of the HLT was to guide the design and evaluation of learning activities aimed at promoting students' understanding of multiplication of fractions.

The HLT was then used to design and evaluate eight learning activities with four different contexts: “Running for Fun”, “Training for Next Year’s Marathon”, “Exploring Playground and Blacktop Areas”, and “Comparing the Cost of Blacktopping”. These contexts were carefully chosen to provide meaningful and engaging situations where students could apply their understanding of fraction multiplication. These activities were administered to a teaching experiment group, allowing researchers to observe and assess students' learning processes and outcomes.

Throughout the study, a conjectured local instruction theory was developed and evaluated using the HLT. This theory represents a set of instructional principles and strategies designed to guide teaching and learning practices. The process of designing, testing, and evaluating the conjectured local instruction theory was described in detail in the pilot study. Based on the findings of the pilot study, the HLT was revised and administered to the teaching experiment group, ensuring the iterative refinement of the instructional approach.

Data were collected from multiple sources, including classroom observations, field notes, and students’completed work. By utilizing multiple sources of data, researchers were able to capture the complexity of students' learning experiences and the mathematical practices that emerged during the fraction multiplication instructional sequence. These data were rigorously analyzed using qualitative methods such as coding and thematic analysis to identify common patterns and themes.

The use of multiple sources of data and the rigorous data analysis process enhanced the trustworthiness and credibility of the research findings. By triangulating data from different sources and using rigorous analytical techniques, the researchers were able to ensure the validity and reliability of their findings (Creswell & Creswell, 2017).

In addition to the use of design research methodology and the HLT, this research study also incorporated elements of formative assessment. Formative assessment is an ongoing and continuous process of gathering evidence of students' understanding and progress, which allows for adjustments and improvements in instruction (Black &

Wiliam, 1998). In this study, formative assessment was used to inform the design of the learning activities and the refinement of the HLT. The researchers continuously gathered data from observations and students' work, analyzed the evidence, and made necessary modifications to the instructional approach based on students' responses and needs.

To ensure the validity and reliability of the research findings, the researchers employed strategies to establish credibility and dependability. Credibility refers to the extent to which the findings accurately represent the participants' experiences and perspectives, while dependability refers to the consistency and stability of the research process and its outcomes (Lincoln & Guba, 1985). In this study, strategies such as member checking, where participants were given the opportunity to review and verify the researcher's interpretation of their experiences, were employed to enhance credibility. Furthermore, the use of an established protocol and clear documentation of the research process ensured dependability.

The researcher also sought to enhance the transferability or generalizability of their findings. Transferability refers to the extent to which the findings of a study can be applied to other contexts or participants (Lincoln & Guba, 1985). In this study, the researcher provided a detailed description of the instructional approach, the learning activities, and the contexts in which they were implemented. This level of detail provides readers with the necessary information to consider whether the findings and instructional approach could be applicable in similar educational settings.

Overall, the research study employed a design research methodology, utilizing the HLT, formative assessment, and multiple sources of data to investigate the understanding of multiplication of fractions among students. The rigorous data analysis process, strategies to establish credibility and dependability, and the provision of detailed information for transferability all contributed to the trustworthiness and credibility of the research findings.

CHAPTER 4

RESULTS AND FINDINGS

In this chapter, the results and findings derived from the pilot and teaching experiments conducted in this study are presented. The primary objective of the investigation was to assess the efficacy of the Hypothetical Learning Trajectory (HLT) developed in Chapter 3. The HLT was utilized as a framework to explore the mathematizing process of fifth-grade students as they progressed in their comprehension of fraction multiplication.

4.1. Pilot Experiment

In this section, the pilot experiment conducted to assess the effectiveness of the eight learning activities and the initial HLT developed in this study is described. The pilot experiment involved the participation of two students, P1 and P2. The results were evaluated for each learning activity and served as a basis for enhancing the initial HLT. The data collection encompassed students' strategies, mathematical reasoning, performance, and behavior throughout the activities. Analyzing the data provided valuable insights into the effectiveness of each learning activity and identified areas with potential for improvement.

4.1.1. Activity 1: Running for Fun

The objective of this activity was to facilitate the development of students' conceptual understanding of fraction multiplication. Through the context of Running for Fun, students were tasked with comparing fractions and establishing connections between fractions and natural numbers. They were encouraged to utilize various strategies, such as repeated addition, a double number line, a distributive strategy, or proportional reasoning, to solve problem 3 in Worksheet 1. This problem required them to determine the total distance Andrew and Bella had run this year.

In the first problem presented in the Worksheet, both students were able to answer the questions accurately. They observed that the fourth water station corresponded to half of the total 8 water stations. The expectation was for them to utilize this information to tackle the second problem, which involved determining whether Andrew and Bella ran a greater distance compared to last year. P1 successfully addressed the question by stating, "Because Bella did $\frac{7}{12}$ and Andrew did $\frac{5}{8}$. So, it is more than a half". P1 showcased an ability to perceive that $\frac{7}{12}$and $\frac{5}{8}$ were greater than one-half based on the visual representation provided in Figure 4.1 below.

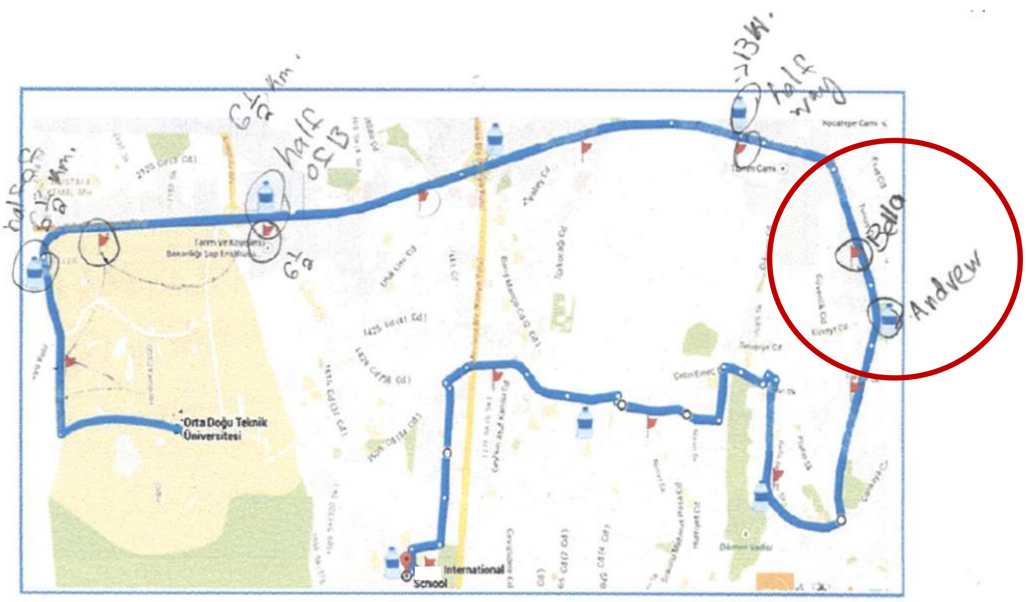


Figure 4.1. P1's marks for Andrew's and Bella's positions

On the other hand, P2 provided a different response. He made reference to the accompanying text, stating that "Andrew and Bella trained together because they would like to get a better one next year". To gain a deeper understanding of P2's interpretation, specific inquiries were posed to him.

Researcher	:	Ok, P2, can you tell me your answer?
P2	:	Yes, because it says that they ran a certain amount the first year and they hoped they'd get a better one next year, and then they did.
Researcher	:	Ok, they want to improve, but how can you know? We have marked the positions of Bella and Andrew. Is it more than a half way?
P2	:	Yes, because this water bottle is the halfway point (pointing out the fourth water bottle in the picture) and Andrew passed it.
Researcher	:	Oh really? Can you mark it and write something that it is a … what? What do you say?
P2	:	Half way
Researcher	:	Alright, so, from here you know that Bella and Andrew?
P2	:	Both got better because they ran half way last year.

This excerpt reveals P2's perception that Andrew's performance surpassed that of the previous year since he surpassed the halfway point, as did Bella.

Subsequently, the primary question, aimed at guiding students towards the concept of multiplication with fractions, was posed. The question entailed determining the total number of kilometers each of them, Andrew and Bella, ran this year. Both P1 and P2 encountered challenges in arriving at the answer. In an effort to elicit responses, the researcher attempted to prompt the students by asking, "how many water bottles in total?". P2 endeavored to convey his thoughts, as evidenced in the following excerpt.

Researcher	:	How many water bottles in total?
P2	:	Eight. Mmm..and then, I think between each water bottles there is a certain number of kilometers.
Researcher	:	So, how can you decide the distance between each water bottle?
P2	:	I can guess but not sure. Maybe two, but that's not true.

Based on the preceding dialogue, it becomes evident that P2 recognized the presence of a specific distance in kilometers between the water bottles. Nevertheless, he encountered difficulty in precisely ascertaining the exact number of kilometers within that distance. The conversation proceeded as follows.

P2	:	This is 13 (pointed out the fourth water bottles). And there is one two three four (counting the number of water bottles after

the fourth water bottles), there is four water bottles and 26 divided by 4?

Researcher : Why 26 divided by 4? Because 26 kilometers is from here (pointed out Orta Doğu Teknik Üniversitesi in the map) to here (pointed out International School in the map)

P1 : How did you get 13? (P1 asked P2)

P2 : Because half of 26 is 13. 26 is the whole thing.

P2 persisted in contemplating the distance between the water bottles and, in a sudden moment of clarity, he observed that the fourth water bottle was strategically positioned at the midway point of the route.

P2 : Ah, I know, 13 divided by 4!. And each is $3\frac{1}{4}$.

Researcher : Yes, and then how can you calculate the length of Andrew?

P2 : It will be 13 kilometers plus $3\frac{1}{4}$ which makes $16\frac{1}{4}$

P2's approach, as demonstrated above, showcases his utilization of proportional reasoning, just as it was hypothesized. Initially, he sought to determine half of 26, resulting in 13. He then proceeded to calculate $\frac{1}{4}$ of 13, which equates to $3\frac{1}{4}$. By decomposing $\frac{5}{8}$ into $\frac{4}{8}$ and $\frac{1}{8}$, P2 deduced that he needed to combine 13 kilometers with $3\frac{1}{4}$ to ascertain $\frac{5}{8}$ of 26 kilometers. Figure 4.2 depicts P2's methodology to determine Andrew's total distance covered.

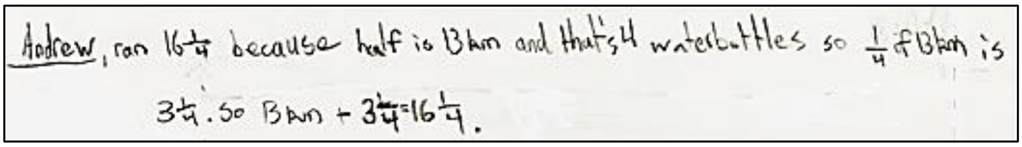


Figure 4.2. P2's answer on how many kilometers Andrew ran

Meanwhile, as P2 continued his pursuit to uncover the number of kilometers Bella ran, the researcher turned to P1 to hear her perspective.

P1	:	So, 13 kilometers is a half of it.
Researcher	:	So, this is 13 kilometers, right? And now, where is the position of Andrew.
P1	:	Andrew is over here (pointed out the fifth water bottles)
Researcher	:	How many water bottles in these 13 kilometers?
P1	:	Four
Researcher	:	And, 4 water bottles for 13 kilometers. So how many kilometers for 1 water bottle (the distance from one water bottle to the next water bottle).
P1	:	Three?
Researcher	:	How can you calculate three? If it is three, then next?
P1	:	6, 9, and then 12.
Researcher	:	And it does not make 13, right? Do you get the idea?
P1	:	No. Mm..So there are 8 water bottles and half of it is 4. So, and then there is 26 kilometers and half of it will be 13.
Researcher	:	And Andrew ran more than half. How can you measure how far Andrew ran?
P1	:	So, it will be three plus something.

As the conversation progressed, the researcher took the opportunity to refresh the students' recollection of converting improper fractions to mixed fractions, recognizing its potential usefulness in solving the problem at hand. P1, although recalling that she had previously learned this concept, found herself momentarily uncertain of the steps involved. To assist her, the researcher presented several examples of improper fractions, and after several attempts, P1 successfully transformed them into mixed fractions. With this refresher completed, attention refocused on the initial quandary of determining Andrew's total distance covered. Figure 4.3 provides an illustration of P1's strategy for calculating Andrew's kilometers run this year.

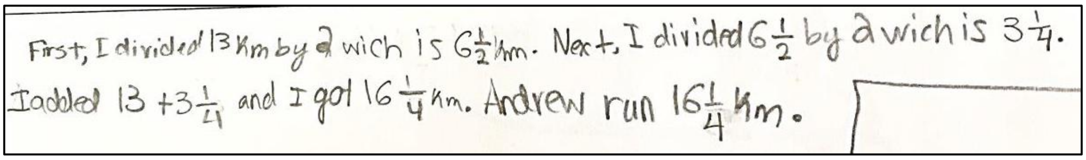


Figure 4.3. P1's answer on how many kilometers Andrew ran

P1's approach, similar to P2's strategy, involved proportional reasoning and decomposing numbers to arrive at the solution. However, there were distinctions between their methodologies. P1 initially divided 13 by 2, resulting in $6\frac{1}{2}$. Continuing her process, she divided $6\frac{1}{2}$ by 2 once again, which equated to $3\frac{1}{4}$, representing the distance between successive water bottles. To determine the total distance Andrew ran this year, P1 added 13 kilometers to $3\frac{1}{4}$.

Moving on to the subsequent problem, where the objective was to ascertain Bella's total distance covered, both P1 and P2 encountered difficulties due to the involvement of seemingly complex fractions. Specifically, they were required to calculate $\frac{7}{12}$ of 26 kilometers. Our conversation with P1 and P2 aimed at encouraging them to utilize the number line as a visual representation to determine the extent of Bella's kilometers run this year.

P2	:	But then Bella's.
Researcher	:	Yes, how about Bella. How many markers are there?
P2	:	There are 12 markers and 7 of it. So, half is $\frac{6}{12}$ and then..
Researcher	:	Half is $\frac{6}{12}$? How can you get that?
P2	:	Because half of 12 is 6.
Researcher	:	Hm..have you heard about number line?
P1 and P2	:	Yes.
Researcher	:	Can you connect the running route with number line?
P2	:	I don't know.
Researcher	:	How about you P1?
P1	:	Hmm..this is running route, and we have fractions on the same position.
P2	:	Oh, how about we strech the running route into a line?
Researcher	:	Yes! We can stretch the running route, as if it's in a straight line.
P1	:	Oh, it's like number line.
Researcher	:	Can you make a number line and show me which one is $\frac{6}{12}$?
P2	:	This is 12 over 12 (pointed out the end of number line and wrote down $\frac{12}{12}$). And now seven twelfths. This is one twelfth (wrote down $\frac{1}{12}$) and then two three four five (pointed out the next lines) and then this is half way and six twelfth (wrote down $\frac{6}{12}$). Um, seven twelfth which is where she is (wrote down $\frac{7}{12}$ to the

		next line after $\frac{6}{12}$) and then eight nine ten eleven and that twelfth over twelfth.
Researcher	:	Ok, now, from here to here how many kilometers? (pointed out the number line made by P2 from 0 to $\frac{12}{12}$)
P2	:	13.
Researcher	:	No, from here to here (pointed out again the number line from 0 to $\frac{12}{12}$).
P2	:	Oh, 26.
Researcher	:	Can you show me?

Next, P2 proceeded by sketching a curved line from 0 to $\frac{12}{12}$ and labeled it with 26 kilometers, as depicted in Figure 4.4. Additionally, P2 made a notation of 13 kilometers, representing half of the total distance, on his number line. At this point, P2 realized the need to determine how many kilometers Bella ran. Recognizing this, the researcher requested that P2 illustrate his approach to calculating Andrew's distance on the number line, resulting in Figure 4.5.

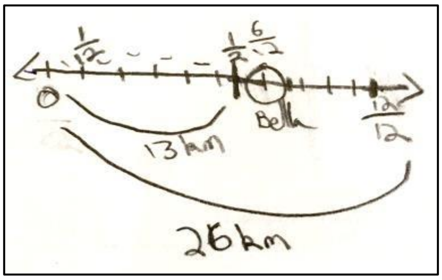


Figure 4.4. P2's double number line to find how many kilometers Bella ran

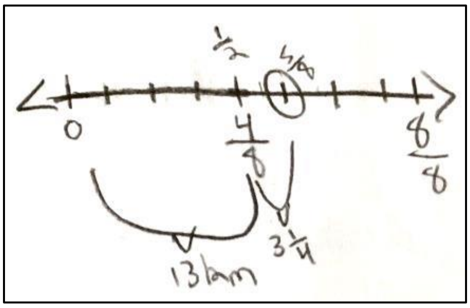


Figure 4.5. P2's double number line to find how many kilometers Andrew ran

Following a reminder regarding P2's approach to determining the distance between water bottles in Andrew's case, P2 had a breakthrough. He realized that to find the distance between markers for Bella's situation, he could divide 26 by 12 or 13 by 6 (representing halfway). Employing the method of converting improper fractions to mixed fractions, P2 showcased his strategy in Figure 4.6, obtaining the outcome $2\frac{1}{6}$. Subsequently, in order to determine the total distance Bella ran in this activity, P2 added 13 to $2\frac{1}{6}$, resulting in $15\frac{1}{6}$.

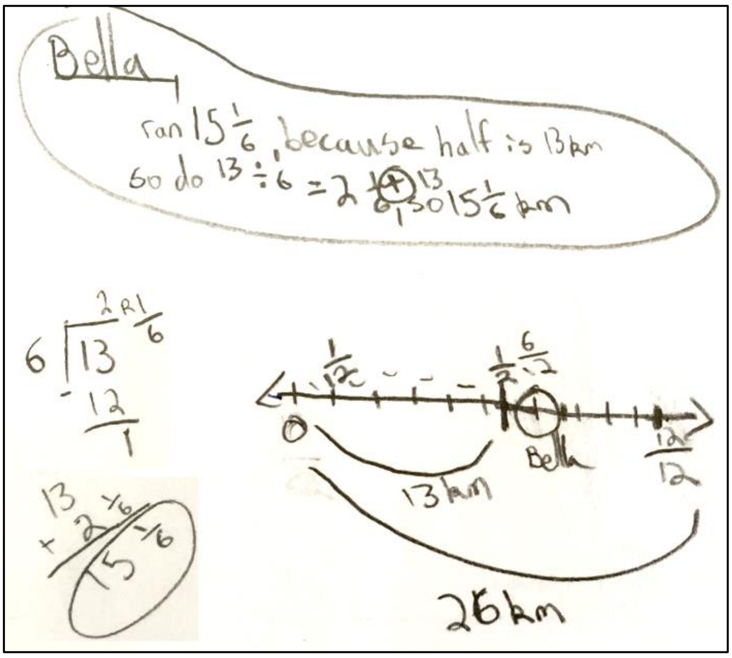


Figure 4.6. P2's answer on how many kilometers Bella ran

P2's solution demonstrates the application of proportional reasoning and the use of a number line as a visual representation of the problem. This particular number line is referred to as a double number line, in accordance with the hypothesis presented in Chapter 3 regarding students' thinking process. Similarly, P1 employed proportional reasoning in her strategy to determine the distance Bella ran in this activity, as depicted in Figure 4.7.

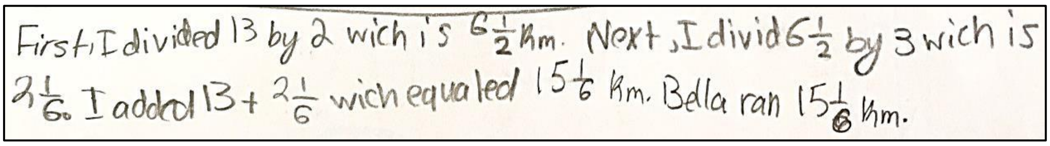


Figure 4.7. P1’s answer on how many kilometers Bella ran

After conducting the activity 1, it became evident that there were some areas of improvement needed for worksheet 1. Specifically, it was suggested to separate the running route from the worksheet to ensure that students could clearly see the water bottles (represented by the signs for “eights”) and the markers (represented by the signs for “twelfths”). Furthermore, it was recommended to engage in a prior knowledge discussion at the start of the lesson, focusing on equivalent fractions and converting improper fractions to mixed fractions.

4.1.2. Activity 2: Math Congress – Running for Fun and Minilesson “Fractions as Operator”

The objective of this activity was to apply the approach utilized by the students to solve problems in Activity 1 to the Minilesson. As observed in the previous activity, the students primarily relied on proportional reasoning and breaking down numbers to solve fraction multiplication. In this activity, they persisted with their previous strategy, despite the researcher's encouragement to use a number line to solve the Minilesson. P2 and P1 encountered no difficulties when solving question number 1 and 2, opting to employ division as demonstrated in Figures 4.8 and 4.9 below.

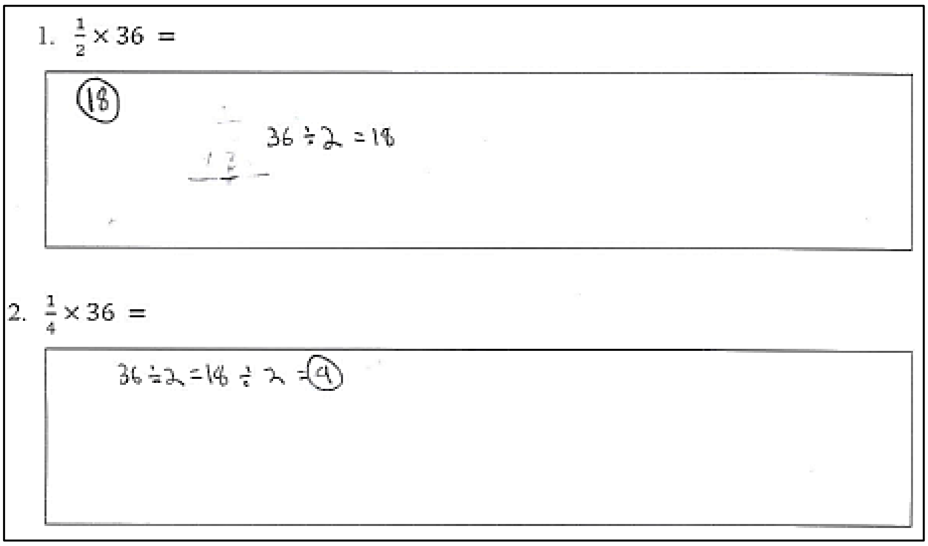


Figure 4.8. P2's strategy to solve question 1 and 2 in the Minilesson

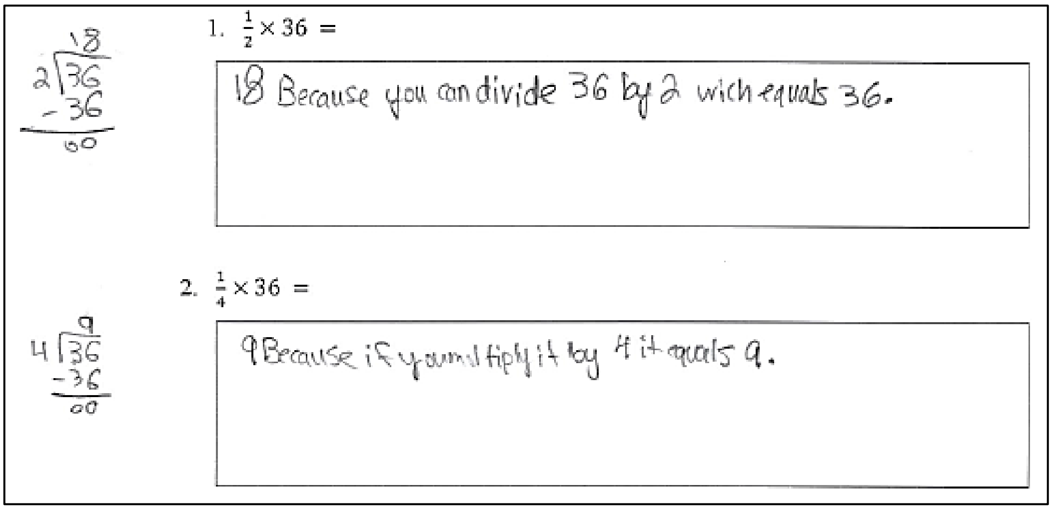


Figure 4.9. P1's strategy to solve question 1 and 2 in the Minilesson

Based on P1's response above, it is evident that she employed the concept of natural number division and had a clear understanding that half of 36 equates to 36 divided by 2. However, when expressing her calculations verbally, she inaccurately wrote, "18 because you can divide 36 by 2 which equals 36". This imprecise explanation was also evident in her answer to question number 2. After engaging in discussions, it became apparent that although P1 understood she needed to divide 36 by 4 to find a quarter of 36, she appeared confused when articulating her calculations in words.

The strategy employed by P1 to solve question number 1 and 2 was also employed to solve question number 3, which involved $\frac{1}{8} \times 36$. Once again, she divided 36 by 8 and incorporated the concept of equivalent fractions to determine the final result of $\frac{1}{8} \times 36$, which amounted to $4\frac{1}{2}$, as depicted in Figure 4.10 below.

Her strategy used to find the solution of question number 1 and 2 was used to solve question number 3, namely $\frac{1}{8} \times 36$. Again, she divided 36 by 8 and she also combined her strategy with the idea of equivalent fraction to find the end result of $\frac{1}{8} \times 36$, namely $4\frac{1}{2}$ as shown in Figure 4.10 below.

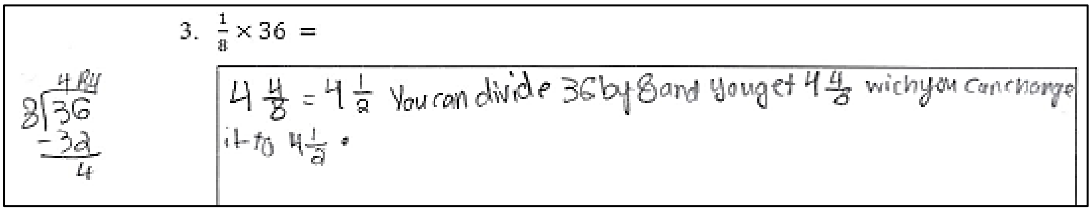


Figure 4.10. P1’s strategy to solve $\frac{1}{8} \times 36$

In contrast to P1, P2 employed a strategy of repeated division, as illustrated in Figure 4.11. When asked to explain his approach, P2 stated that he aimed to find “a half of a half of a half of 36”, leading him to utilize repeated division.

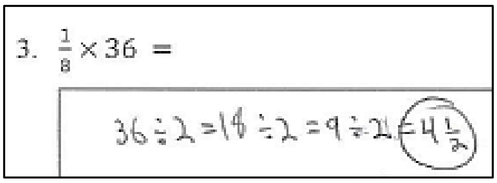


Figure 4.11. P2’s strategy in finding the solution of $\frac{1}{8} \times 36$

The discussion progressed to question number 4, where the students encountered challenges in solving the problem of $\frac{5}{8} \times 36$. To guide the students in resolving this difficulty, the researcher encouraged them to consider using the concept of a number line, similar to their experience in solving the Running for Fun problem in Worksheet 1. The researcher began by rewriting the results of $\frac{1}{2} \times 36$, $\frac{1}{4} \times 36$, and $\frac{1}{8} \times 36$ on the board, aiming to prompt the students to recognize that they can utilize the outcome of $\frac{1}{8} \times 36$ to find $\frac{5}{8} \times 36$. A number line was drawn to visually represent the fractions and connect them with whole numbers, as depicted in Figure 4.12.

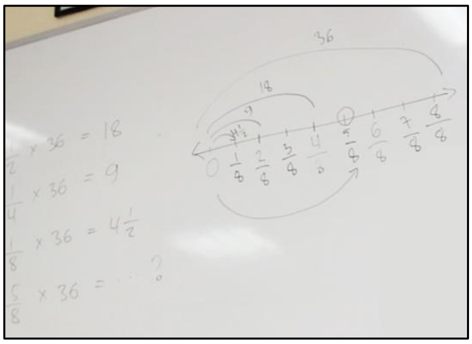


Figure 4.12. Double number line to show how to find $\frac{5}{8} \times 36$

The following dialogue captures the interaction between the researcher and the students regarding the depiction presented in Figure 4.12.

Researcher	:	Half of $\frac{8}{8}$ is $\frac{4}{8}$ and half of $\frac{4}{8}$ is $\frac{2}{8}$ and here is $\frac{1}{8}$ and here is $\frac{3}{8}$, then here $\frac{6}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$. (Together with the students, researcher completed the number line of $\frac{8}{8}$ fraction.)
Researcher	:	So, now actually we make a new context, this is 36 kilometers or something. (The researcher made an arrow from 0 to $\frac{8}{8}$ and wrote 36 above of it.)
Researcher	:	And how about from here to here? (drew arrow from 0 to $\frac{4}{8}$).
P2	:	18

Researcher	:	Actually, we calculate this, 18, a half of 36. And the, we also calculate this, $\frac{1}{8}$ of 36. What is it?
P2	:	$4\frac{1}{2}$
Researcher	:	So, one jump of $\frac{1}{8}$ is $4\frac{1}{2}$. We want to know from 0 to $\frac{5}{8}$. Remember, one jump $4\frac{1}{2}$. How many times we need to add to reach $\frac{5}{8}$?
P2	:	Mm..5 times?
Researcher	:	Yes

One of the students was invited to solve the problem on the white board. P2 stepped forward willingly to solve 5 times of $4\frac{1}{2}$. P2 utilized proportional reasoning, as hypothesized in the HLT, as illustrated in Figure 4.13.

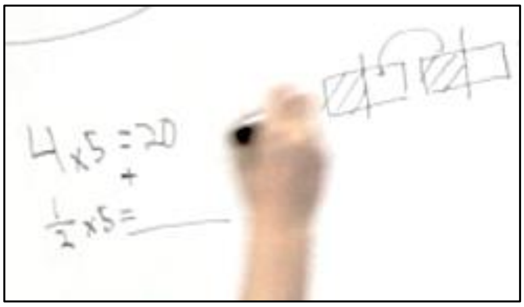


Figure 4.13. Proportional reasoning strategy to find $5 \times 4\frac{1}{2}$

In contrast, P1 approached the problem using the concept of fractions in pizza to find $5 \times \frac{1}{2}$, as depicted in Figure 4.14.

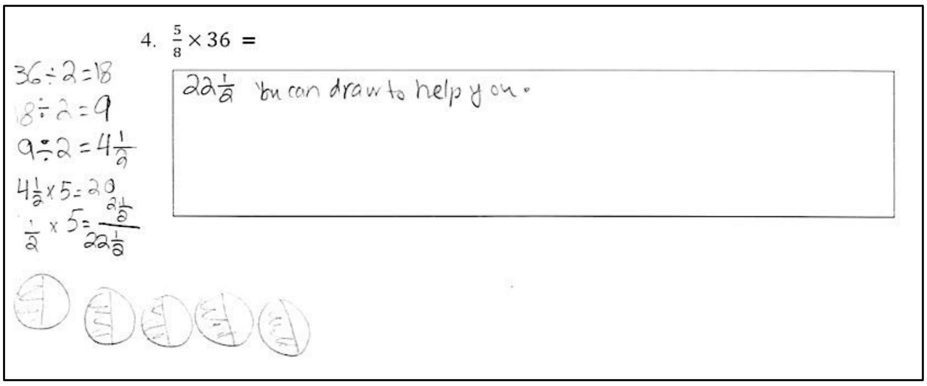


Figure 4.14. P1's strategy to solve $\frac{5}{8} \times 36$

Before proceeding to solve problem number 5, which involved $\frac{7}{8} \times 36$, the researcher provided a reminder to the students about converting mixed numbers to improper fractions by presenting several examples. Once it was confirmed that the students grasped the concept of converting mixed numbers, the researcher prompted the students to verify the relationship between numbers in the fraction multiplication of $\frac{1}{2} \times 5 = \frac{5}{2}$. P1 observed that the result featured the same numbers and remarked, "The 1 went away and the 5 replaced it". The researcher then explained to the students that 5 has the same meaning as $\frac{5}{1}$, illustrating this concept through multiple examples in Figure 4.15 until the students understood the process of multiplying fractions by a natural number.

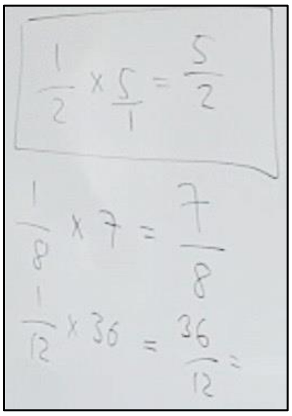


Figure 4.15. Samples of multiplication of fraction with natural number

The students proceeded to solve problem number 5 and 6 utilizing the same approach, as depicted in Figures 4.16 and 4.17. An additional exercise, question number 7 (i.e., $\frac{3}{4} \times 15$), was introduced.

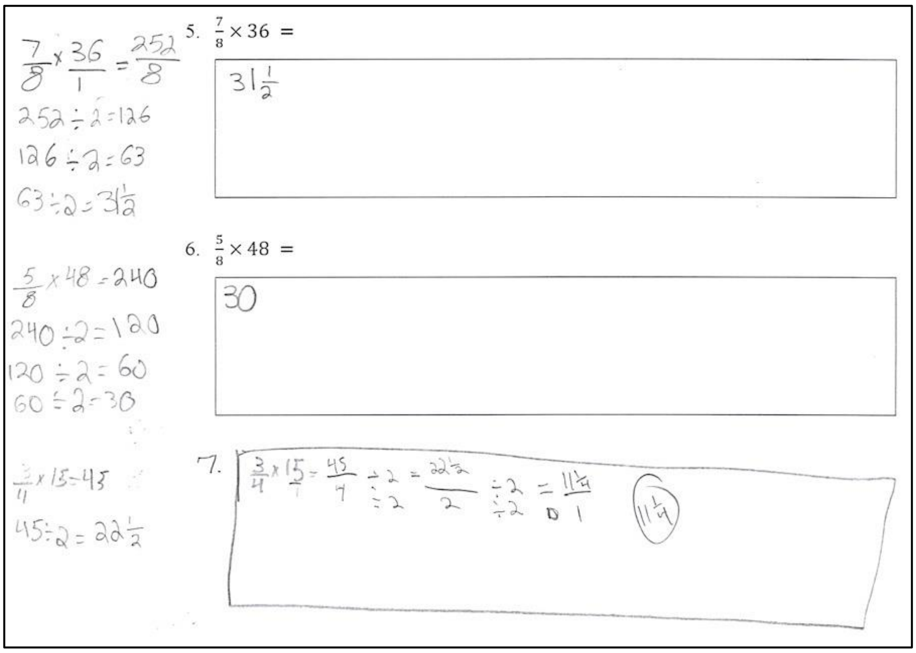


Figure 4.16. P1's strategy to solve $\frac{7}{8} \times 36$, $\frac{5}{8} \times 36$ and $\frac{3}{4} \times 15$

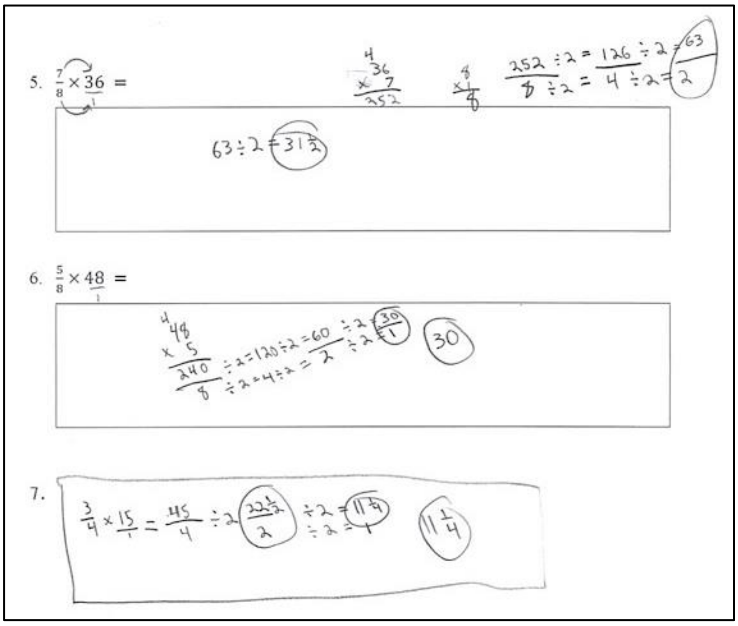


Figure 4.17. P2's strategy to solve $\frac{7}{8} \times 36$, $\frac{5}{8} \times 36$ and $\frac{3}{4} \times 15$

Following their engagement with Activity 2, it became evident that the students generated several significant concepts associated with the multiplication of fractions, such as the application of the distributive property and proportional reasoning. The minilesson provided an opportunity for further reflection on these strategies. Additionally, it was identified that there is a need to supplement the exercise with more questions for the students.

4.1.3. Activity 3: Training for Next Year's Marathon

The objectives of this activity were to involve the students in utilizing landmark fractions and partial products when multiplying fractions with a whole number, as well as to encourage exploration of multiplication and division involving a natural number and a fraction, and the connections between these operations. In this particular activity, the numbers displayed on the chart were deliberately selected to prompt discussions on the relationships between multiplication and division with natural numbers, and the application of these operations with rational numbers. The initial three problems involved the use of natural numbers for both the number of minutes and the number of circuits completed. Both P2 and Bella demonstrated competence in solving these initial three problems, as evident in the following excerpt.

Researcher	:	For Alex, how many minutes he completed?
P2	:	120
Researcher	:	Circuits of track completed is 4. So, in 120 minutes Alex can complete 4 tracks. And, you need to find the rate, rate is minutes per circuit.
P1	:	So, do you divide?
Researcher	:	Minutes per circuit
P1	:	30?
Researcher	:	Yes!

Based on the aforementioned discussion, P1 demonstrated an understanding of the relationship between the number of minutes and the circuits of the track completed. The instructions clearly stated that the rate is measured in minutes per circuit. Both P1 and P2 recognized that the word "per" indicates division. The students proceeded to determine the rates of Ethan and John. When it came to finding the minutes for Benjamin, P2 sought clarification from the researcher regarding the validity of his strategy, as evidenced in the following excerpt.

P2	:	So now here, you do 20 times 1 (pointing out the circuit completed by Benjamin and his rate)
Researcher	:	Why do you do that? Why do you do multiplication?
P2	:	Cause the opposite of divided. When you divide here, so you need to multiply here, to go backward.
P2	:	For Olivia, 18 times $\frac{1}{2}$, so, like 18 times 1 but half of 1? So, it's 9?
Researcher	:	How do you think? Is it correct?
P2	:	I do not know, because it has to be more?
Researcher	:	Are you sure? Why it has to be more?
P2	:	Because she ran some of the track so she has to add some. But she can't run negative.
Researcher	:	You need to think again, from the first problem, look at the pattern. Alex finished the running for 4 tracks in 120 minutes. So, how many minutes for 1 track? 30. So here, the rate itself is how many minutes to complete 1 track. And now for Olivia, she could complete 1 track in 18 minutes. So, how many minutes if she could only complete the half of the track?
P2	:	Mm..9?
Researcher	:	How you got it?
P2	:	Because half of 18 is 9.

In the above discussion, it becomes apparent that P2 encountered cognitive conflict when questioned by the researcher about his certainty that Olivia takes 9 minutes to complete half of the track. P2 realized that Olivia did not finish the entire track, making it impossible for her to have a negative time. The researcher attempted to prompt P2 to reassess the pattern from the first problem and implicitly suggested that the "rate" refers to the number of minutes required to complete a full track. P2 then adopted the strategy of halving to determine that a half of 18 minutes is equal to 9 minutes.

In contrast, P1 did not face any difficulties as she recognized the pattern in the problems. She understood that to determine the rate, the number of minutes should be divided by the number of circuits completed, and to find the minutes, the rate should be multiplied by the number of circuits. The following excerpt illustrates P1's attempt to guide P2 as he attempted to find the minutes for James.

P1	:	P2, remember how we changed this (pointing out $1\frac{1}{2}$) into improper fraction?
P2	:	Yes
P1	:	So, make this into improper fraction and then multiply by 20 and see what you get.

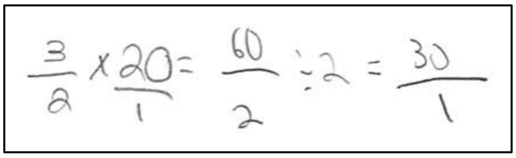


Figure 4.18. P1's suggestion to change the mixed number into improper fraction

Based on the preceding discussion and as depicted in Figure 4.18, P1 employed the strategy of multiplying fractions that was utilized in the previous activity. To provide the students with more opportunities to comprehend the process of multiplying a mixed number with a fraction, an additional problem is planned to be included in this activity.

4.1.4. Activity 4: Math Congress – The Marathon Training Results and Minilesson "Fractions as Operator"

During this activity, the students were given the opportunity to discuss their ideas and strategies for solving the problems in Activity 3 before proceeding with the exercise in the mini-lesson. P1 took the initiative to explain her strategy, which involved multiplying the rate and the number of circuits to determine the number of minutes. For example, when she multiplied 30 (the rate) by 4 (the number of circuits) for Alex, she obtained 120 (the number of minutes). P1 applied the same strategy for Benjamin, multiplying 20 (the rate) by 1 (the circuit) to find the number of minutes. She conveyed her approach by stating, "So, it takes 20 minutes to complete 1 round." The discussion suggests that P1 was able to recognize the relationships in the data and utilize them to solve the problems. When solving the multiplication of a fraction with a whole number (e.g., Olivia, Emma, Isabella, and James), P1 employed the strategy of multiplying the numerators together and the denominators together. This strategy was derived from the previous activity.

Meanwhile, P2 utilized a different strategy for multiplying a fraction with a whole number, as seen when the researcher asked him to explain how he solved the problem involving Emma ($\frac{1}{4} \times 20$), as indicated in the following excerpt.

Researcher	:	How about Emma, P2?
P2	:	5? Oh, because 20 and a fourth and if you divided by 4 it makes 5.
Researcher	:	Wait, you said 20 divided by 4? Let me take P2’s idea.

The researcher proceeded to write “$\frac{1}{4} \times 20$” on the board and attempted to interpret P2's explanation as “20 divided by 4”. Another example was presented on the board, involving Olivia ($\frac{1}{2} \times 18$), as shown in Figure 4.19. However, P2 expressed doubt about whether his approach would yield the same result as the strategy for multiplying a fraction with a whole number, stating, “But, if you do it the way you told us (multiplying the numerator with the numerator and the denominator with the denominator), do you end up with the same answer?” P1 agreed that it would indeed result in the same answer and subsequently demonstrated the process of multiplying a fraction with a whole number on the board.

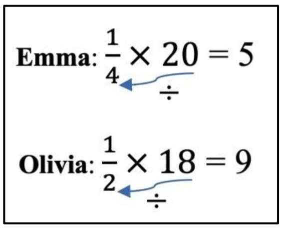


Figure 4.19. P2’s idea of multiplying fraction with natural number

The discussion then shifted to the case of Isabella, who only ran $\frac{3}{4}$ of the circuit at a rate of 20. The researcher drew a circle representing the complete track and indicated where Isabella had stopped. She then asked the students to identify the position on the track corresponding to 5 minutes. P2 correctly understood that 5 minutes represented $\frac{1}{4}$ of the track. He observed that every $\frac{1}{4}$ of the track took Isabella 5 minutes to run, as shown in Figure 4.20. P2 explained, "Isabella only got here (pointing), she did not go all the way, so she stopped here. Five minutes times three equals fifteen minutes". From his explanation, it is evident that P2 used partial products and regarded $\frac{1}{4}$ of 20 minutes (5 minutes) as a reference fraction.

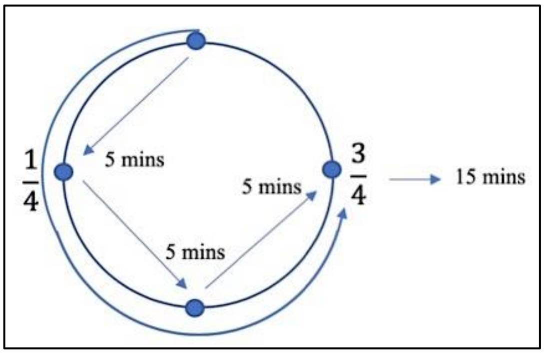


Figure 4.20. P2's idea by using landmark fractions to make partial product

Prior to the students continuing with Worksheet 4, the researcher posed a question about who the fastest runner was among the nine individuals training for the marathon. P2 contended that "whoever has the lower rate is better". He compared Alex and Ethan and deduced that Ethan was the faster runner because his rate (20 minutes per circuit) was lower than Alex's rate (30 minutes per circuit). This discussion aligned with the conjecture of students' thinking. The students demonstrated an ability to recognize the relationships in the numbers they were working with and thus reached the conclusion that a runner could have a faster rate because they ran for a shorter duration.

Following the examination of the students' problem-solving strategies in Activity 3, they proceeded to work on Minilesson 4: Fractions as Operator. The students encountered no challenges in solving the problems presented in the Minilesson. However, three additional questions will be incorporated into the Minilesson to provide the students with further practice in multiplying fractions with whole numbers.

4.1.5. Activity 5: Exploring Playgrounds and Blacktop Areas

In this activity, the students were given the task of solving a problem that involved multiplying fractions within the context of area. A scenario was presented to the students, involving the construction of blacktop areas on playgrounds in two identical empty lots. This context was introduced to help the students develop a deeper understanding of multiplication of fractions. The problem required them to determine if one of the lots had more blacktop and explain their strategy for comparing the blacktop areas.

At the beginning of the activity, P1 read the problem aloud and asked for the definition of blacktop. P2 then explained that blacktop referred to asphalt, like what they had in their school's park for playing basketball. Once the problem was fully read, the researcher facilitated an initial discussion.

Researcher	:	So, there are 2 gardens, Botany and Gulhane, and, can you tell me what's the story about?
P2	:	One of them is going to be covered with the blacktop.
Researcher	:	One of them or two of them?
P2	:	One of them.
P1	:	Actually, isn't both of them? Both of them, Botany and Gulhane are having a little bit of blacktop.
Researcher	:	For Botany Garden, $\frac{3}{4}$ of the lot will be devoted to a playground and then?
P2	:	$\frac{1}{4}$ will be blacktopped.
P1	:	No, $\frac{2}{5}$ will be blacktopped.
		Now, how big the garden? What is the measurement?
Researcher	:	50 meters by 100 meters
P1	:	Ok, now can you draw 50 meters by 100 meters?
Researcher	:	(Both students drew rectangle.)

From the passage provided, it is evident that P1 acknowledged that only a portion of both Botany and Gulhane gardens would be covered in blacktop. This observation is crucial in guiding the students towards comprehending the concept of part of a whole, as only a section of the gardens will be blacktopped. The discussion continued, and the students were prompted to illustrate the fraction $\frac{3}{4}$. Both students had no difficulty in representing $\frac{3}{4}$ on the rectangle they had created. Initially, they divided the rectangle into four parts and shaded three of them. However, the students encountered challenges when tasked with finding $\frac{2}{5}$ of the playground that would be covered in blacktop. To address this, the researcher attempted to stimulate the students by posing the question of whether the blacktop would be located inside or outside the boundaries of the playground, as elaborated below.

Researcher	:	We have Botany garden, 50 meters by 100 meters, and then $\frac{3}{4}$ of that we want to make a playground. And, from that playground, we want to cover it with blacktop. So, is the blacktop inside the playground or outside the playground?
P2	:	So, here's the playground (showing the $\frac{3}{4}$ shaded area of his drawing), and then we will find $\frac{2}{5}$ from this shaded area. I will not include this part. So, this is the $\frac{2}{5}$.

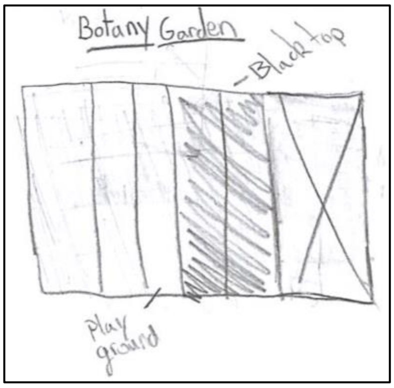


Figure 4.21. P2's first drawing to represent $\frac{2}{5}$ of $\frac{3}{4}$ (the shaded area)

Based on the previous discussion and P2's drawing, it is evident that P2 understood the task of finding $\frac{2}{5}$ from the shaded area of $\frac{3}{4}$. However, there was confusion as P2 mistakenly mixed the divided parts, as depicted in Figure 4.21. Eventually, P2 arrived at a fraction of $\frac{2}{6}$ for the blacktop, after canceling out $\frac{1}{4}$ and dividing the remaining $\frac{3}{4}$ into five parts and shading two of them to represent $\frac{2}{5}$. This issue was mentioned in the HLT, where students struggled with vertical divisions of fourths and fifths. Although they identified the overlapping section, determining the fractional part proved challenging.

Observing the students' difficulties, the researcher provided a hint that they could divide the parts both horizontally and vertically. By dividing the rectangle into four to find $\frac{3}{4}$ horizontally, they could then cut the rectangle vertically to find $\frac{2}{5}$. Additionally, the researcher informed the students that they could shade the parts in the opposite manner, facilitating the identification of the blacktop area as the section shaded twice. Finally, the students were able to determine the blacktop sections. However, they encountered further challenges when attempting to represent the blacktopped area as a fraction, as illustrated in the subsequent discussion.

Researcher	:	So, show me which part of the blacktop. Can you make a fraction of it?
P1	:	$\frac{2}{5}$.
Researcher	:	How many small boxes now we have?
P2	:	15? 5..5..5..(P2 used his pen to show that there are 5 boxes in 3 rows)
P1	:	18.
P2	:	Oh, 20.
Researcher	:	Why?
P2	:	Because I count on the top too.
Researcher	:	Yes, do not forget, we need to include the whole. The idea of fraction is part of a whole. So what fraction is it?
P2	:	$\frac{6}{20}$.

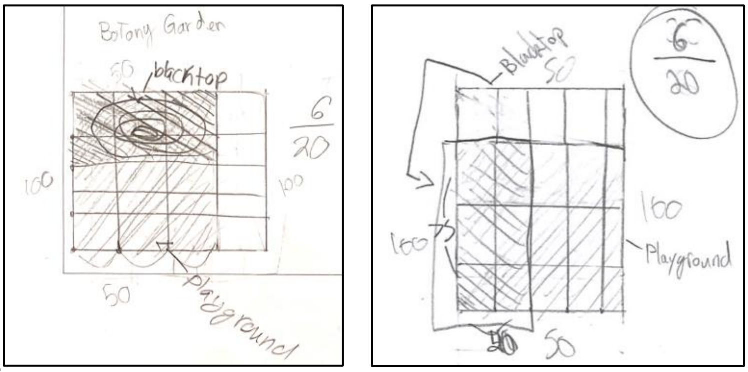


Figure 4.22. (left to right) P1's and P2's drawings representing $\frac{2}{5}$ of $\frac{3}{4}$

Using the same approach, the students encountered no difficulties in determining the blacktop area of Gulhane garden. Both P1 and P2 discovered that the fraction representing the blacktopped area of Gulhane garden was the same as that of Botany garden, specifically $\frac{6}{20}$. This provided an answer to the initial question regarding whether one garden had more blacktop space than the other.

Regarding the second question regarding their rationale for the correct conclusion they reached, P1 explained, "if you draw it and put it next to each other you see the squares that are shaded are the same. The area is the same but the shading is not because they switch the place". From P1's explanation, it can be inferred that she recognized the similarity of the blacktopped areas in both gardens solely by observing the fractional parts, without directly calculating the actual area. Additionally, she emphasized that the areas were equivalent but not congruent due to the switching of their positions.

In contrast to P1, P2 attempted to determine the blacktopped areas of both gardens, as depicted in Figure 4.23, and discovered that both gardens had a blacktopped area of 1,500 m^2.

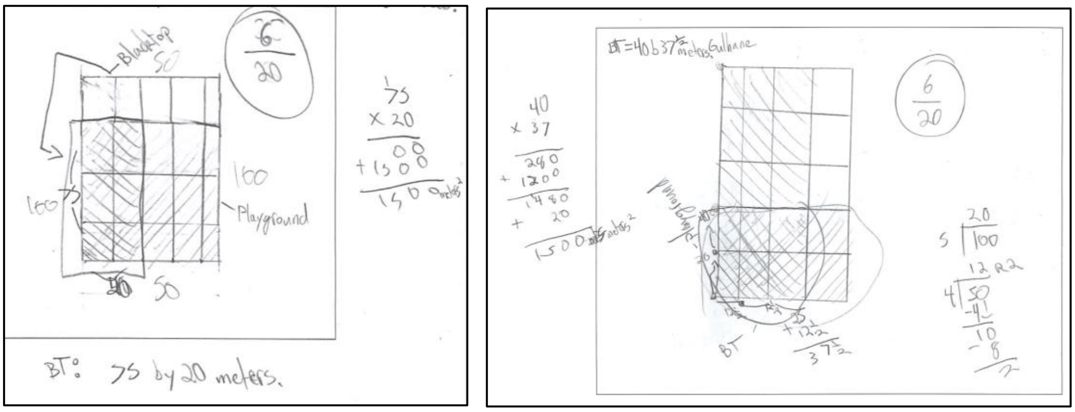


Figure 4.23. Strategy in finding the blacktopped areas by considering the dimension of the lots

P2’s approach aligns with the hypothesis that students may use the dimensions of the lots, which are 50 meters by 100 meters, to compare the blacktopped areas of the two gardens.

Regarding the problem itself, as evidenced by the preceding discussions, the students initially encountered difficulties in comprehending the context. However, with some guidance, the students were able to devise their own strategies to solve the problem. To provide further clarity and aid in the teaching experience, an additional piece of information will be included in the problem, specifying the names of the gardens. This will underscore the fact that there are two distinct gardens that are to be blacktopped.

4.1.6. Activity 6: Math Congress – Minilesson “Multiplication of Fractions”

Prior to delving into the Minilesson of Activity 6, the students discussed the strategies they employed to solve the problem in Activity 5. The researcher aimed to guide the students in understanding the concept conveyed by the word "of" in the given context. The example of the blacktopped area of Botany garden was used as a point of reference in the discussion. In the previous activity, the students had determined that the fractional part representing the blacktopped area of Botany garden was $\frac{6}{20}$. They arrived at this fraction by multiplying $\frac{2}{5}$ by $\frac{3}{4}$, which represented $\frac{2}{5}$ of $\frac{3}{4}$ of the playground. Recognizing the connection between these fractions, the students grasped that the word “of” indicated multiplication with a fraction (as illustrated in Figure 4.24).

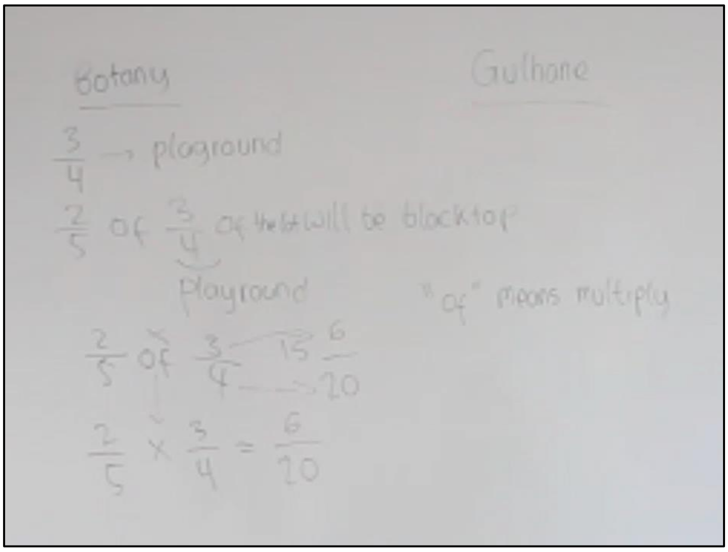


Figure 4.24. Illustrating the interpretation of the word “of” as indicating the multiplication of a fraction

Additionally, during the discussion, the students made another significant discovery – the results of calculating $\frac{2}{5}$ of $\frac{3}{4}$ (for Botany garden) and $\frac{3}{4}$ of $\frac{2}{5}$ (for Gulhane garden) were identical. This led the students to realize that the commutative property also applies to fractions, specifically, $\frac{2}{5} \times \frac{3}{4} = \frac{3}{4} \times \frac{2}{5}$.

Moreover, the students also gained insight into the array model from a previous activity. They observed that the blacktopped section of the playground was identical in both gardens, represented as “6 boxes out of 20”, with the shaded portions being equal in both cases.

After discussing the strategies and concepts underlying the problem in Activity 5, the students proceeded to solve the string problem in the Minilesson. Having already developed strategies for multiplying fractions, they encountered no difficulties in completing this activity. However, the researcher concluded that it would be beneficial to include two additional questions in the Minilesson, providing the students with more practice and allowing them to further explore the relationships between the numbers within the fraction.

4.1.7. Activity 7: Comparing the Cost of Blacktopping

In this activity, the objective for the students was to utilize the array model to solve a multiplication of fractions problem. They were reminded of their previous investigation, where they concluded that both the Botany and Gulhane gardens had an equal blacktopped area of $\frac{6}{20}$. They were then prompted to redraw the representation of the previous problem on the whiteboard, as depicted in Figure 4.25.

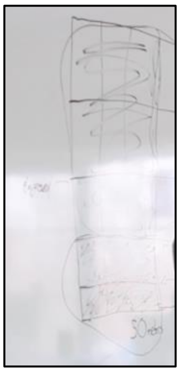


Figure 4.25. Drawing the fractional parts of blacktopped areas of Botany and Gulhane gardens

After refreshing their memory about the previous problem, one of the students was invited by the researcher to read aloud the problem of Comparing the Cost of Blacktopping. Initially, the students struggled to comprehend the purpose of the percentage in the first problem (the cost of blacktopping in the Botany garden), which stated, “the cost of blacktopping in the first lot is $9 per square meter, but the contractor will offer to do it at 80% of that price”. The researcher facilitated an initial discussion and informed the students that there would be a discount, with the people in the Botany garden area only needing to pay 80%. P2 proposed the idea that they needed to calculate 80% of $9, as discussed further in the following dialogue.

Researcher : For first problem, you will pay 80%.
P2 : So, 80% of \$9?
Researcher : Yes, you will pay 80% of \$9 for what?
P2 : Per square meter.
Researcher : And how many square meters the area?
P2 : 1,500 m^2. Oh, so we have to figure out what is 80% of \$9.
Researcher : Why?
P2 : Because that is how much you need to pay per square meter and then there is 1,500 m^2, then you times (multiply) that (the result of 80% of \$9).

When the students attempted to find 80% of \$9, they encountered another challenge. The researcher reminded them about the meaning of the word "of" from a previous activity, where they had learned that it signifies multiplication. However, both students mistakenly arrived at 720% as the result of multiplying 80% by 9. The researcher then reminded the students how to convert 80% into a fraction. Both students had no difficulty in converting the percentage into a fraction, and they even simplified it (from $\frac{80}{100}$ to $\frac{8}{10}$). However, P2 struggled when multiplying the fraction by 9. P1 came to P2's aid and employed the strategy for multiplying a fraction by a whole number that she had acquired from activity 2 (refer to Figure 4.26).

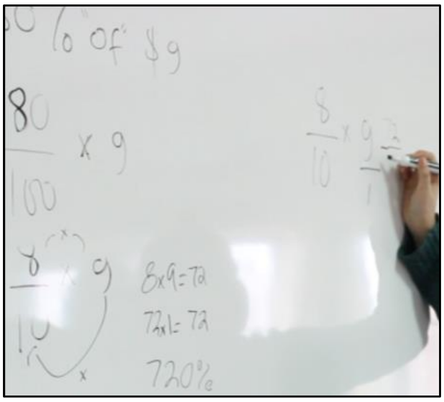


Figure 4.26. Students tried to find 80% of \$9

Once the students determined the result of 80% of \$9, which is $7\frac{1}{5}$, they realized that this was not the total cost of blacktopping for the Botany garden. Instead, it was the discounted price per square meter offered by the contractor. They needed to multiply this price by the blacktopped area, which was 1,500 m^2. To calculate $7\frac{1}{5} \times 1{,}500$, the students broke down $7\frac{1}{5}$ into 7 and $\frac{1}{5}$. First, they multiplied 7 by 1,500, and then $\frac{1}{5}$ by 1,500, as illustrated in Figure 4.27. The students applied the distributive property, as hypothesized in the HLT, for the calculation of $7\frac{1}{5} \times 1{,}500 = (7 \times 1{,}500) + (\frac{1}{5} \times 1{,}500)$. Both P2 and P1 arrived at the total cost of blacktopping for the Botany garden, which was \$10,800.

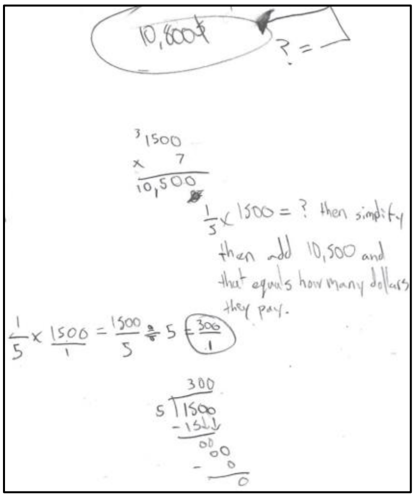


Figure 4.27. P2's strategy in finding $7\frac{1}{5} \times 1{,}500$

Using the same method, the students were instructed to calculate the cost of blacktopping in the Gulhane garden. They translated the phrase "90% of \$8" into the fraction $\frac{9}{10} \times 8$ and determined that the price of blacktopping per square meter for the Gulhane garden is $7\frac{1}{5}$. To find the total cost of blacktopping in the Gulhane garden, they needed to multiply $7\frac{1}{5}$ by 1,500 m^2. However, P2 realized that there was no need to recalculate and that the previous calculation could be used. He concluded that the cost of blacktopping in both gardens would be the same, as the price per square meter was \$$7\frac{1}{5}$, as shown in Figure 4.28.

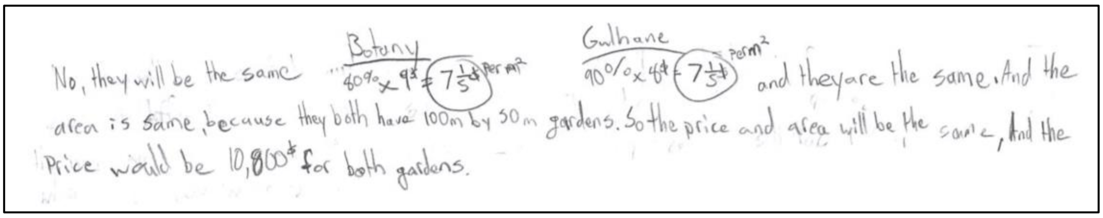


Figure 4.28. P2's reason that the costs of blacktopping for both gardens are the same

Once the problem was solved and the students determined that the costs of blacktopping for both gardens were equal, the researcher guided them to explore the concept behind the problem, which is discussed further in the following excerpt.

Researcher	:	From your strategy, we have $\frac{8}{10} \times 9$ for Botany and $\frac{9}{10} \times 8$ for Gulhane.
P2	:	Then you end up with the same answer.
Researcher	:	Anything you can see from the numbers?
P2 and P1	:	It is commutative property!
P1		
Researcher	:	You switch places of 8 and 9
	:	Yes, we actually interchanging numerators here.

From the above information, it is evident that the students understood the strategy of rearranging numerators to multiply by a fraction, which is based on the commutative property. This strategy would prove useful for them to solve a series of problems in activity 8.

4.1.8. Activity 8: Math Congress – Minilesson "Interchanging Numerators"

The aim of this activity was for students to solve a series of fraction multiplication problems. The numbers chosen were carefully selected to allow students to apply the commutative property of fraction multiplication and explore the concept of interchanging numerators. Both participants in the study demonstrated competence in solving these problems during the minilesson.

However, in order to further strengthen students' comprehension and engagement, the researchers decided to replace question number 7 in the activity. Instead of $\frac{3}{5} \times \frac{2}{3}$, they substituted it with $\frac{7}{9} \times \frac{3}{14}$. This change was made to create a problem that would be more accessible and relatable to the students, making it easier for them to grasp the concept.

4.1.9. General Conclusion of the Pilot Experiment

The initial HLT trial was an important step towards students' understanding of multiplication of fractions, a topic that is often challenging for students, especially when moving beyond simple natural number multiplication. The findings of the pilot experiment offered interesting insights into students' thinking and their initial understanding of the concept. It was clear that students had some prior knowledge and strategies that they could draw on when solving problems involving multiplication of fractions. For example, they understood that fractions represent a relation and they were able to use the distributive, associative, and commutative properties that hold for natural numbers, also applied for rational numbers.

Moreover, students were able to use skip-counting and repeated addition to find a fraction of a whole, and they were able to use landmark fractions to make partial products. They also demonstrated some understanding of the standard algorithm for multiplication of fractions, using the ratio of the product of numerators to the product of denominators, and interchanging numerators (or denominators) to simplify first when multiplying.

Despite these initial understandings, it became clear that students needed further support and guidance in developing a deep and meaningful understanding of multiplication of fractions. There were gaps and misconceptions in their thinking that hindered their ability to solve more complex problems involving multiplication of fractions. Therefore, it was necessary to revise parts of the initial activities to better address these misconceptions and to provide additional opportunities for students to engage with the concept in a meaningful way.

The findings that were gathered from the observations that took place during this pilot experiment have been discussed in the previous sections. It played a crucial role in informing the development of the revised HLT and worksheets. The findings from the observations provided valuable insights into the obstacles students faced when learning multiplication of fractions, as well as the prior knowledge required for successful engagement with the activities. As a result, the HLT and worksheets were carefully examined and improved based on the conclusions drawn from the pilot experiment.

The revised HLT (Table 4.1) was designed to address the obstacles identified during the pilot experiment and to scaffold students' learning of multiplication of fractions. The inclusion of landmark fractions and the use of manipulatives to support students' visualizing of the concept were key aspects of the revised HLT. Additionally, opportunities for students to work collaboratively and discuss their thinking were also emphasized in order to promote the exchange of ideas and the development of mathematical language and communication skills.

Moreover, the worksheets were improved to provide students with additional practice and scaffolding opportunities related to the concepts learned in the HLT. These improvements included prior knowledge required, and additional problems that gradually increase in difficulty.

The revised HLT and worksheets were refined to better facilitate the learning process and improve the engagement of students. The next research cycle, which was the teaching experiment phase, aimed to test the efficacy of the revised HLT and assess the effectiveness of the modifications made based on the conclusions acquired from the pilot experiment.

Overall, the pilot experiment was a critical step in refining the learning activities and improving the initial HLT. The insights gained from the pilot experiment were used to revise the HLT and tailor the learning activities to promote understanding and engagement among the students. The findings from the pilot experiment were also used to inform the subsequent teaching experiment, which evaluated the effectiveness of the revised HLT and learning activities on a different group of students.

Table 4.1. Revised HLT for learning multiplication of fractions after pilot experiment (phase 1)

<table>
<tr><th colspan="2">Activities
• Running for Fun
• Math Congress
• Minilesson 1: Fractions as Operators</th></tr>
<tr><td>Learning Goals and Processes</td><td>• Students will develop several big ideas related to multiplication with fractions.
• Students will share their ideas in solving problem especially in decomposing numbers and using partial products.
• Students will revisit their strategies discussed in the math congress and use the idea of decomposing numbers and partial products to solve the string of Minilesson 1.</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>• Fractions represent a relation
• The whole matters
• To maintain equivalence, the ratio of the related numbers must be kept constant
• The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers</td></tr>
<tr><td>Prior Knowledge Required</td><td>• Equivalent fractions
• How to change improper fractions to mixed fractions, and vice versa</td></tr>
<tr><td>Model for Fraction</td><td>(Double) number line</td></tr>
</table>

Students' Possible Strategies
• Students notice that the fourth water station means half of 8 water stations. • Students notice that $\frac{7}{12}$ and $\frac{5}{8}$ is more than a half. • Students notice that Andrew ran better than last year because he passed the halfway point, so did Bella. • To find how many kilometers Andrew ran, students use the idea of proportional reasoning and decomposing number. • First, they try to find a half of 26, which is 13, then find $\frac{1}{4}$ of 13 which is $3\frac{1}{4}$. As they decompose $\frac{5}{8}$ into $\frac{4}{8}+\frac{1}{8}$, means that he needed to add up 13 kilometers with $3\frac{1}{4}$ to find $\frac{5}{8}$ of 26 kilometers. • Another strategy which uses the idea proportional reasoning and decomposing number is that students divide 13 first by 2 which made $6\frac{1}{2}$, then divided $6\frac{1}{2}$ again by 2, which is $3\frac{1}{4}$, to find the distance from one water bottle to the next water bottle. Then, they added 13 kilometers with $3\frac{1}{4}$ to find the length of Andrew ran this year. • Students use proportional reasoning and decomposing numbers to solve multiplication of fraction with natural number in Minilesson 1.
Students' Obstacles
• Students first struggle to connect the fraction (representing the position of market/water station) with natural number (length of running route). • Students struggle to determine how many kilometers the distance between the water bottles. • Students struggle to work with unfriendly fraction, namely to find $\frac{7}{12}$ of 26 kilometers.
Activities • **Training for Next Year's Marathon** • **Math Congress** • **Minilesson 2: Fractions as Operators**

Learning Goals and Processes	• Students will use landmark fractions and partial products when multiplying a fraction by a natural number.

<table>
<tr><td></td><td>

- Students will explore multiplication and division (a natural number by a fraction) and the relationship between the operations.
- Students will share their ideas in solving problem in Activity 3 especially about the patterns on the chart and ideas related to multiplication and division with fractions.

</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>

- Fractions represent a relation
- The whole matters
- To maintain equivalence, the ratio of the related numbers must be kept constant
- The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers

</td></tr>
<tr><td>Prior Knowledge Required</td><td>

- Equivalent fractions
- How to change improper fractions to mixed fractions, and vice versa

</td></tr>
<tr><td>Model for Fraction</td><td>Ratio table and (double) number line</td></tr>
<tr><td colspan="2">Students' Possible Strategies</td></tr>
<tr><td colspan="2">

- Students may find that in order to divide (by fractions), they are multiplying (by the multiplicative inverse of fraction) to find the rate.
- Some students may notice the numbers relationships and use proportional reasoning to complete the chart. Proportional reasoning is the heart of this problem in which equivalence will be maintained when the ratio kept constant. For instance, the students might see since Alex ran twice as many minutes as Elizabeth and completed twice as much distance as Elizabeth, then his rate is the same as Elizabeth's rate: 30 minutes for one circuit.
- Partial quotients may also be used, for instance for the case of Isabella, she did $\frac{3}{4}$ of the circuit in 15 minutes. Then, $\frac{3}{4}$ will be divided into 3 pieces and each piece was 5 minutes. So, the whole is 20 which we add 15 and 5, because that is $\frac{3}{4}+\frac{1}{4}$.
- Some may use the relationship of multiplication and division, for instance in the case of Elizabeth, the students may find the minutes first namely 2×30 is 60. Alex ran twice as far. Then they will think about $4 \times ? = 120$, and the result will be 30, too.
- Some students may use the double number line model as they try to connect with the previous context, Running for Fun.

</td></tr>
</table>

- Others may use the chart as ratio table in which the arrows will show the relations of the rates.

Students' Obstacles

- Student may face difficulty to see in the case of Olivia and Emma as their rates are differ. It is expected that the students will draw the track and mark out the fractions on the circuit if they struggle with Olivia and Emma's case.
- Students may struggle to work with unfriendly fraction, i.e., $2\frac{3}{4}$ (circuit of track completed) of 30 (rate) ($2\frac{3}{4} \times 30$).

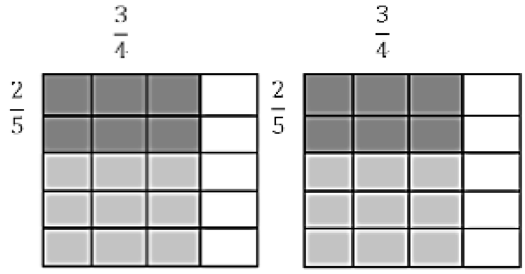
Activities

- **Exploring Playgrounds and Blacktop Areas**
- **Math Congress**
- **Minilesson 3: Multiplication of Fractions**

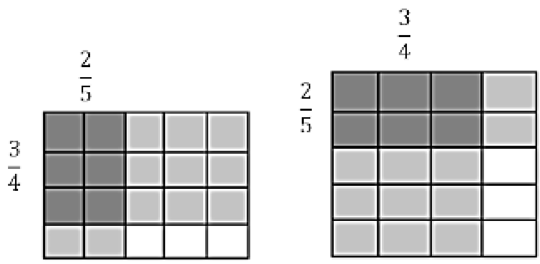
Learning Goals and Processes	• Students will solve the problem related to multiplication of fractions. • Students will share their ideas in solving problem in Activity 5.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Prior Knowledge Required	• Representing fraction in drawing as 'part of a whole'
Model for Fraction	Array / area model

Students' Possible Strategies

- Students will cut both lots fourths vertically and fifths horizontally and then shade or color the parts in which indicate the blacktopped area, $\frac{3}{4}$ of $\frac{2}{5}$ or $\frac{2}{5}$ of $\frac{3}{4}$. With this strategy, the ratio of the array of the blacktopped area (3×2) to the array of the lot (4×5). So, the blacktopped area of two lots will congruent.



- Students will cut one lot into fourth horizontally then shade $\frac{3}{4}$ indicating the playground, then marking fifths of that area vertically and shade $\frac{2}{5}$ of it the show the blacktop area. Similar way for the other lot, students will first cut the lot into fifths horizontally then shade $\frac{2}{5}$ indicating the playground, then marking fourths of that area vertically and shade $\frac{3}{4}$ of it the show the blacktop area.



- Students perhaps cut the fifths or fourths vertically (or horizontally).
- Students may use the dimensions of the lots, which is 50 meters × 100 meters. This strategy may give result 37.5 meters × 40 meters, and 75 meters × 20 meters. Using this strategy, the students will find that the areas are equivalent but not congruent.

Students' Obstacles

- When students cut the fifths or fourths either vertically (or horizontally), they will find overlapping part but they might struggle to determine the fractional part.
- Students may be confronted with the problem of finding a means to compare areas that are similar but not congruent.
- Students may struggle to work with unfriendly fraction, i.e., $\frac{6}{20}$ (fraction representing the blacktop area) of 1,500 m^2 (total area of one lot).

<table>
<tr><td colspan="2">Activities
• Comparing the Cost of Blacktopping
• Math Congress
• Minilesson 4: Interchanging Numerators</td></tr>
<tr><td>Learning Goals and Processes</td><td>• Students will use array model to solve the multiplication of fractions problem.
• Students will extend their investigation with the commutative property of multiplication of fractions to percentages and decimals.
• Students will use interchanging numerators (or denominators) to derive the product of two fractions.
• Students will share their ideas in solving problem in Activity 8 especially related to the commutative property and associative property of multiplication with fractions, percentages and decimals.</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>• Fractions represent a relation
• The whole matters
• The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers</td></tr>
<tr><td>Prior Knowledge Required</td><td>• How to convert percentage into fraction</td></tr>
<tr><td>Model for Fraction</td><td>Array / area model</td></tr>
<tr><td colspan="2">Students' Possible Strategies</td></tr>
<tr><td colspan="2">• Before comparing the cost, the students will calculate the area of blacktopping for each lot.
• The calculation depends on their drawing from previous activity, whether they will calculate $\frac{3}{4}$ of 50 and $\frac{2}{5}$ of 100; or $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100. In calculating $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100, students will probably not have difficulty than calculating $\frac{3}{4}$ of 50. In calculating $\frac{3}{4}$ of 50, students might start with what they know, for instance by first find $\frac{1}{2}$ of 50 (=25) and $\frac{1}{2}$ of 25 (=12.5) which is same as $\frac{1}{4}$ of 50, and then they can calculate $\frac{3}{4}$ of 50 as 37.5 meters.
• Students might find the commutative properties that underlies the relationship of both blacktopping in two lots:</td></tr>
</table>

37.5 meters × 40 yards = 75 meters × 1,500 yards = 1,500 square meters.

- After students find the blacktopping areas of two lots, students may multiply the area with \$9 per square meters and offers 80% of the price, students might multiply \$9 by 1,500 to determine the full price of blacktopping which is \$13,500; then to find 20% of \$13,500, students will use landmark of fraction in which 20% equal to $\frac{1}{5}$ and then calculate $\frac{1}{5}$ of \$13,500 (the discount, by dividing by 5, \$2,700); and subtract that discount to get the final price of \$10,800. Similarly, to find the price of blacktopping in other lot uses \$9 per square meters and $\frac{1}{10}$ to calculate the discount.
- Some students will directly include the discount in the calculation and use decimal or fraction forms to show the percentages (i.e., 80% = 0.8 = $\frac{8}{10}$; 90% = 0.9 = $\frac{9}{10}$). Then, calculate $0.8 \times \$9\ \times\ 1{,}500\ square\ meters$ for the cost of first blacktopping area and calculate $0.9 \times \$8\ \times\ 1{,}500\ square\ meters$ for the cost of second blacktopping area. To find the calculation, the students might decompose the percentages or fractions through associative property:

$$\left(8 \times \frac{1}{10}\right) \times 9 \times 1{,}500$$

$$= 8 \times (\frac{1}{10} \times 9) \times 1{,}500$$

- Other students may think that there is no need to include the area as they found it is equivalent. They may just use $0.8 \times \$9$ and $0.9 \times \$8$. In this case, they might be challenged about why 80% of 9 = 90% of 8. It is expected that the students will find the associative property underlies the equivalence:

$$8 \times \left(10 \times \frac{1}{100}\right) \times 9$$

$$= 9 \times \left(10 \times \frac{1}{100}\right) \times 8$$

Students' Obstacles

- Student may directly multiply 80% by \$9 as they connect the meaning of word "of" from previous activity as multiplication.
- Students have a difficulty in converting percentage into fraction.
- Students may struggle to work with unfriendly fraction, i.e., $7\frac{1}{5}$ (80% of \$9, $\frac{80}{100} \times \$9$) of 1,500 m^2 (the area of the lot).

4.2. Teaching Experiment

At the teaching experiment phase, we tried out the revised sequence activities with three students (S1, S2, and S3). The contexts used in the sequence of activities of this study, namely Running for Fun, Training for Next Year's Marathon, Exploring Playgrounds and Blacktop Areas, and Comparing the Cost of Blacktopping will be used to develop the analysis of the teaching experiment phase. Additionally, the stages of mathematizing processes, including modeling, symbolizing, generalizing, formalizing, and abstracting, will be applied to guide the analysis and answer the research questions. Along the way of the analysis, students' obstacles in learning fraction multiplication will also be explained in the following sections.

4.2.1. Number Line as a Model for Learning Multiplication of Fraction with Natural Number - Running for Fun Context

As per the first tenet of Realistic Mathematics Education (RME), contextual problems serve as starting points for students to explore and derive mathematical concepts. In this study, the context of Running for Fun was chosen to provide students with an opportunity to develop a deep understanding of multiplication of fractions and to construct a number line as a model for solving fraction problems.

Before engaging with the problem, students were asked to share their prior knowledge about fractions. They correctly identified fractions as representing parts of a whole that are divided into equal parts. To further assess their understanding, the researcher instructed the students to draw representations of various fractions, as depicted in Figure 4.29. This exercise allowed the researcher to gauge the students' proficiency in visualizing and representing fractions.

Additionally, the researcher inquired if the students had learned about the simplification of fractions. This information was crucial to determine whether the students possessed the prerequisite knowledge required to tackle the problem at hand. Understanding if the students were familiar with simplifying fractions would be useful in guiding the instructional approach and ensuring that each student's unique needs were addressed.

By taking into account the students' prior knowledge and providing them with a meaningful context, RME aims to engage students actively in the learning process. This approach encourages students to construct their understanding of fractions and their operations through real-world scenarios and hands-on activities.

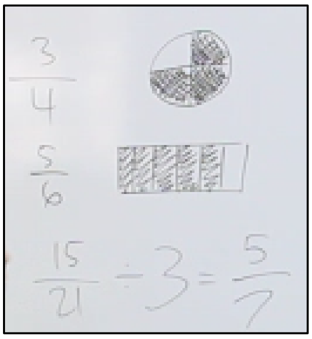


Figure 4.29. Prerequisite knowledge about fractions

Furthermore, the students were presented with fractions addition problems involving the same denominators. As the students had previously learned about fractions addition, they encountered no difficulties in solving these problems. The ability to perform fractions addition is considered a prerequisite skill that will be utilized in solving the Running for Fun problem.

In addition to fractions addition, the students were also given questions pertaining to the positions of Andrew and Bella last year and their current positions this year. Prior to answering these questions, the students were instructed to mark the positions on a map given to them. Remarkably, all students successfully marked the positions without any difficulty.

Upon analyzing the marked positions, the students observed that both Andrew and Bella had run halfway last year. Specifically, Andrew had stopped at the 4^{th} water station, while Bella had stopped at the 6^{th} marker. By closely examining the positions of the water stations in the provided picture, the students effortlessly answered the first and second problem. They astutely noted that "there are 8 water stations and half of 8 is 4". Furthermore, they confidently explained that Andrew and Bella had run a greater distance this year compared to last year.

The students' ability to accurately locate the positions and draw insights from the diagram showcases a solid understanding of fractions and spatial relationship concepts. This understanding is crucial in effectively solving the subsequent problems related to the Running for Fun context.

S2 and S3 engaged in a discussion regarding the distances covered by Bella and Andrew. They both asserted that last year, they had run half of the route. However, this year, Bella ran $\frac{7}{12}$ of the route while Andrew ran $\frac{5}{8}$, indicated by the positions marked by the water stations. Meanwhile, S1 took a different approach. She initially aimed to determine the fractions representing halfway for both Bella and Andrew. Employing the concept of equivalent fractions, she established that $\frac{6}{12}$ represented halfway for Bella last year, and $\frac{4}{8}$ represented halfway for Andrew. Reflecting upon this information, she inferred that Bella and Andrew had run farther this year, as they had stopped at $\frac{7}{12}$ and $\frac{5}{8}$, respectively, which exceed the halfway point.

Subsequently, the students were presented with the next question, which required them to calculate the total distance covered by both Andrew and Bella this year. S1 opted to directly determine the distances for each marker and water station, as depicted in Figure 4.30.

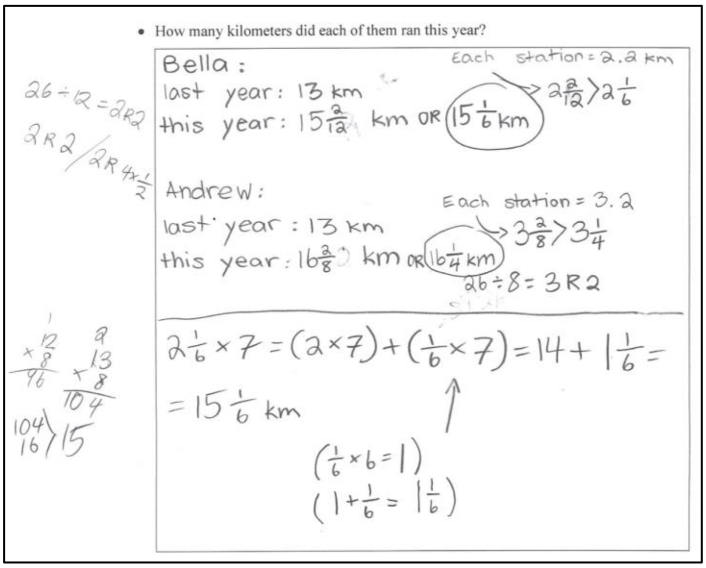


Figure 4.30. S1's strategy in determining how far Andrew and Bella ran

In order to determine the distance covered by Bella, S1 employed a calculation involving the length of each marker and each water station. She multiplied $2\frac{1}{6}$, representing the length of each marker, by 7. To perform this multiplication, S1 cleverly utilized the distributive property. She decomposed $2\frac{1}{6} \times 7$ into $(2 \times 7) + (\frac{1}{6} \times 7)$. This decomposition facilitated the calculation and showcased S1's ability to employ strategic thinking. Notably, S1 employed a unique strategy when confronted with the calculation of $\frac{1}{6} \times 7$. She recognized that $\frac{1}{6} \times 6$ yields 1 and deduced that to find $\frac{1}{6} \times 7$, she needed to add the result of $\frac{1}{6} \times 6$ with 1. This strategy aligns with the conjecture of the students' thinking in the HLT, which suggests that by utilizing the distributive property, students can decompose fractions to simplify calculations. S1 also applied this same strategy when calculating the distance covered by Andrew.

Turning to S2 and S3, the researcher encouraged them to employ a double number line as a model to represent the fractions. For S2, the researcher prompted her to determine the distance, in kilometers, between the water stations and markers. S2's strategy is visually depicted in Figure 4.31.

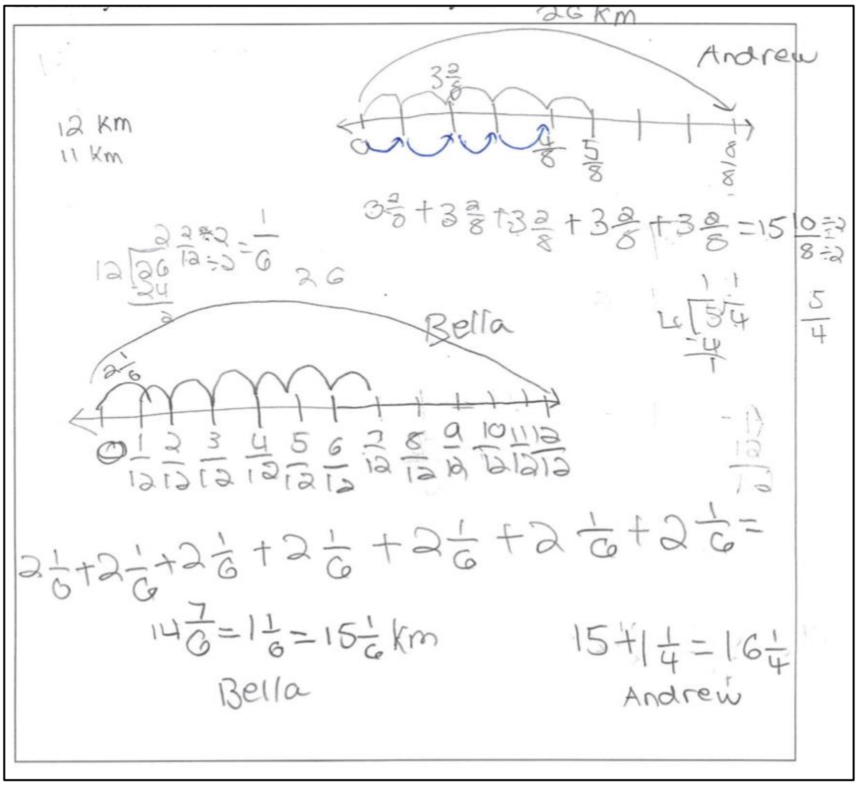


Figure 4.31. S2's strategy in determining how far Andrew and Bella ran

As seen in Figure 4.31, it can be observed that S2 employed a specific approach to determine the distances between the water stations (in Andrew's case) and the markers (in Bella's case). S2 divided the length of the marathon route, 26 km, by 8 (for the water station) and by 12 (for the markers). Subsequently, she employed repeated addition to ascertain the distances traveled by Bella and Andrew. S2's strategy aligns with the HLT, which suggests that students can utilize the double number line concept to demonstrate repeated addition.

An intriguing strategy was employed by S3. Instead of considering the entire length of the marathon route (26 km), S3 opted to utilize the length of the halfway route (13 km) to determine the distances between the water stations and the markers. This unique approach is depicted in Figures 4.32 and 4.33.

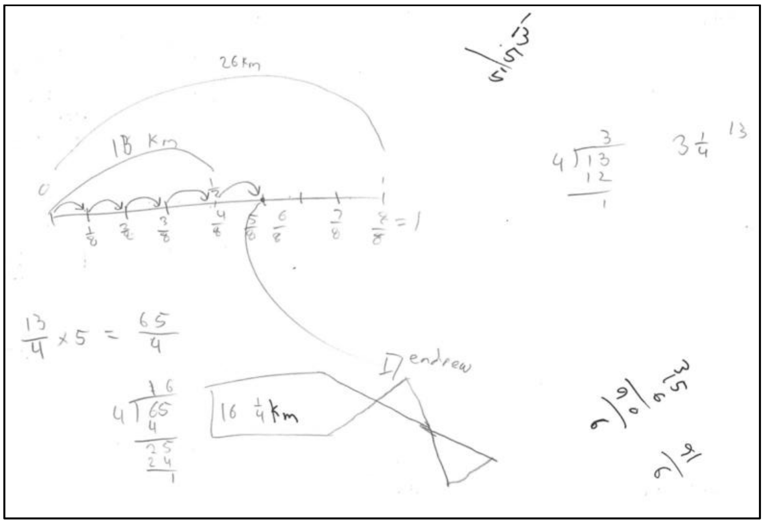


Figure 4.32. S3's strategy in determining how far Andrew ran

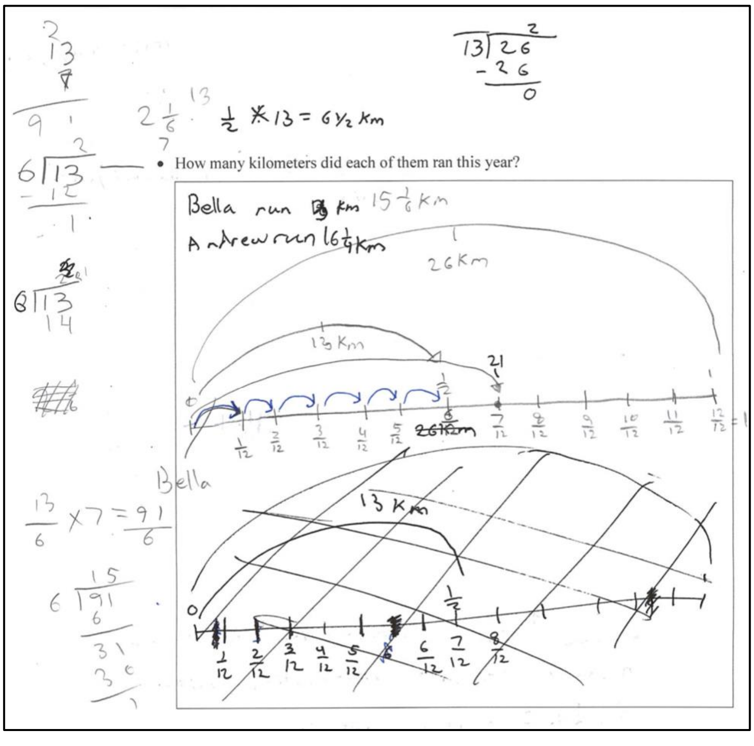


Figure 4.33. S3's strategy in determining how far Bella ran

As seen in Figure 4.33, S3 showcased a distinct approach to calculate the distance covered by Andrew. Recognizing that Andrew only completed half of the marathon route and stopped at the 4th water station (representing a distance of 13 km), S3 divided this distance by 4. This division yielded $3\frac{1}{4}$ km, which represents the distance between each water station. However, to determine how far Andrew ran, S3 employed a multiplication of fractions. Multiplying the distance between each water station ($\frac{13}{4}$ km) by the number of water stations Andrew passed (5), S3 found that Andrew ran a total distance of $\frac{65}{4}$ km. This fraction can be further simplified to $16\frac{1}{4}$ km.

Interestingly, S3 applied the same strategy to determine the distance covered by Bella. Initially dividing the half-way distance of 13 km by the number of markers (6), S3 obtained the distance between the markers, which amounted to $\frac{13}{6}$ km. Multiplying this value by the number of markers Bella passed (7), S3 determined that Bella ran a total of $\frac{91}{6}$ km. After simplifying this fraction, the distance covered by Bella was calculated to be $15\frac{1}{6}$ km.

These strategies implemented by S3 were not originally included in the existing HLT. However, their inclusion has proven to be invaluable for enriching knowledge and fostering opportunities for further research.

During the subsequent meeting, the researcher initiated a math congress, in line with the fourth tenet of RME, which advocates for an interactive learning process. This approach allowed the students to actively participate in their learning by engaging in discussions and debates to further develop their ideas. The focus of this math congress was the Running for Fun problem that had been previously explored by the students. Its purpose was to facilitate and enhance the students' understanding of utilizing the number line as a model, with the ultimate goal of comprehending the meaning of fractions and progressing to the concept of multiplying fractions.

In this math congress, the researcher specifically requested S3 to elucidate his strategy for determining the distance traveled by Bella. S3 asserted that he employed the number line as a model since he envisioned the running route as an extended line. To demonstrate his approach, S3 was asked to illustrate the number line on the whiteboard and indicate the position where Bella stopped. This position coincided with the 7th marker out of the total 12 markers, which can be expressed as a fraction: $\frac{7}{12}$.

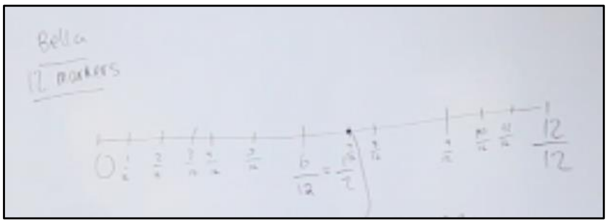


Figure 4.34. S3 drew the number line representing the running route and marked the position of Bella

As depicted in Figure 4.34, the student utilized the number line as a visual representation of the running route. This practice aligns with the second principle of RME, which emphasizes the effective implementation of models and symbols to facilitate learning and problem-solving endeavors (Treffers, 1991a; Gravemeijer, 1994).

The act of drawing the running route on a number line allowed the students to create a tangible visual model. This model played a crucial role as a bridge between the tangible experience of running and the more abstract concept of a number line. By successfully envisioning their running routes on the number line, the students actively engaged in constructing their mathematical understanding. This constructive process involved a gradual transition from a concrete representation to a symbolic one. Through this progression, the students developed an association between the drawn running paths and the corresponding numerical values on the number line.

Employing models and symbols as a result of student participation in problem-solving activities has significant implications for fostering a deeper conceptual understanding. It enables students to establish meaningful connections between their concrete experiences and abstract mathematical concepts, such as number lines. By actively constructing their understanding through the use of models and symbols, students can cultivate a more comprehensive and adaptable mathematical knowledge base.

Following that, the researcher proceeded to inquire about the primary problem at hand. S1 promptly responded by stating that the task involved determining the total distance that Bella had run this year.

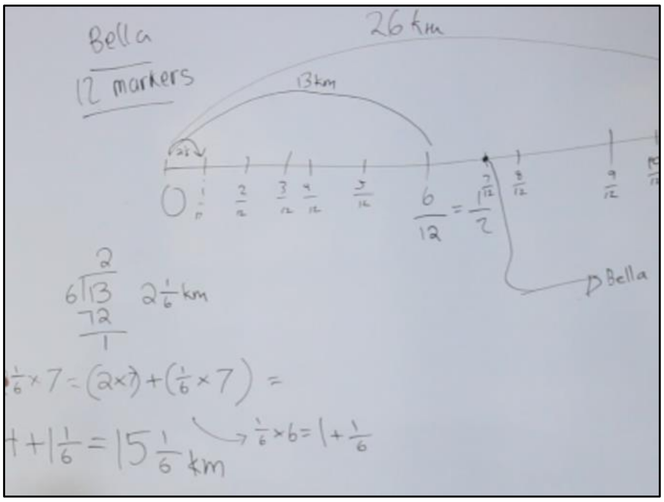


Figure 4.35. S1 explained to her peers how to find how far Bella ran

The subsequent excerpt portrays the dialogue that transpired between the researcher and the students, elucidating the content depicted in Figure 4.35. This conversation aimed to delve deeper into the students' understanding and explore their insights regarding the problem-solving process.

S1 : So, before we find how many kilometers is $\frac{7}{12}$, we need to find $\frac{1}{12}$. But before that, we will find half of 26 km, which is 13.

		And now, because we have 6 parts here, we need to divide 13 by 6, and that would be $2\frac{1}{6}$ km.
Researcher	:	Great, so what does $2\frac{1}{6}$ km mean?
S1	:	It shows the length of 1 part, from 0 to $\frac{1}{12}$ and then $\frac{1}{12}$ to $\frac{2}{12}$ and so on. And then, to get $\frac{7}{12}$ we can just add $2\frac{1}{6}$, $2\frac{1}{6}$, $2\frac{1}{6}$, and so on, or we can just multiply.
Researcher	:	How can you get the idea of multiplication?
S1	:	We need to multiply by 7 since she stopped at the 7th marker and the distance between each marker is $2\frac{1}{6}$. Now we have $2\frac{1}{6} \times 7$. But that would be difficult to multiply. So, separate the fraction and that would be: $2\frac{1}{6} \times 7 = (2 \times 7) + (\frac{1}{6} \times 7)$ For some people $(\frac{1}{6} \times 7)$ may be difficult too. So, if we have 6 equals parts then $\left(\frac{1}{6} \times 7\right) = 1 + \frac{1}{6}$ So, finally: $2\frac{1}{6} \times 7 = (2 \times 7) + \left(\frac{1}{6} \times 7\right) = 14 + 1\frac{1}{6} = 15\frac{1}{6}$
Researcher	:	Alright, thank you S1. S2 and S3, how about your answers? Do you have different answer as S1's?
S2 and S3	:	No, we have the same answer as hers.
Researcher	:	Alright, now let's check S1's strategy, as we can see, S1 separated $2\frac{1}{6} \times 7$ to calculate the multiplication of fraction with natural number, namely $(2 \times 7) + (\frac{1}{6} \times 7)$. Do you know what property it is?
All students	:	No
Researcher	:	Yes, now you learn how to make partial product in order to easily multiply the fraction $(2\frac{1}{6})$ with natural number (7). This property called "distributive property".

Based on the conversation and the accompanying picture, it is evident that the students engaged in a self-directed learning process, particularly when confronted with the task of multiplying "unfriendly" fractions. The term "unfriendly fractions" refers to fractions with complex or non-standard denominators, which can make multiplication more challenging (Johnson, 2020). In this context, the students were faced with the need to multiply fractions with different and potentially complicated denominators.

Despite the difficulty posed by these unfriendly fractions, the students showcased their ability to effectively devise strategies to overcome the challenge. One such strategy was demonstrated by S1, who employed a method of fraction decomposition. This approach involved breaking down the fractions into simpler parts or unit fractions, which in turn enabled the students to more easily manipulate and compute the multiplication.

S1's strategy of decomposing the fractions ultimately led the students to explore and utilize the distributive property as a means to simplify the computation. By distributing the multiplication across the various decomposed fractions, the students were able to combine the resulting products and arrive at the final solution.

Conversely, S2 and S3 opted for a repeated addition strategy to solve the problem. This approach involved adding the fraction repeatedly, the number of times equivalent to the numerator, to arrive at the desired product. While different from S1's decomposition strategy, this repeated addition method also provided a valid and effective approach to solving the problem involving unfriendly fractions.

In light of these varying strategies, the researcher recognized the educational value in promoting peer learning and encouraged S2 to explain her strategy to her peers in Figure 4.36. This exchange not only fosters a deeper understanding of the problem-solving process but also promotes collaboration and the sharing of diverse strategies among the students. By engaging in such peer discussions, students can learn from each other and gain insights into alternative problem-solving techniques (Jones et al., 2018).

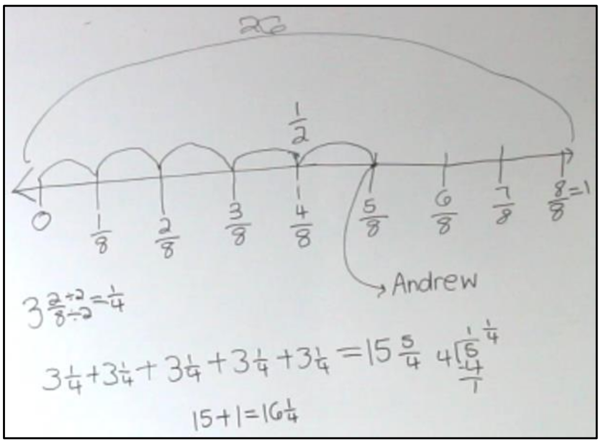


Figure 4.36. S2 showed her peers her strategy to find how far Andrew ran

Figure 4.36 showcased the utilization of repeated addition as a strategy to determine a fraction of a whole. As the discussion unfolded, the students recognized the existence of multiple strategies and models that could be employed to solve the marathon investigation. They generated various significant concepts related to multiplying fractions, including the distributive property and repeated addition.

Building on the students’ progress, the researcher further engaged them by prompting them to identify any patterns that emerged from multiplying fractions with natural numbers. This invitation followed their successful completion of the Bella and Andrew case. The researcher rephrased the multiplication of fractions using the final results, encouraged the students to convert improper fractions into proper fractions, and directed their attention to the pattern evident in the excerpt provided, complemented by Figure 4.37.

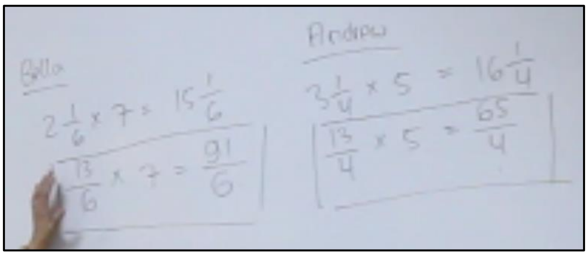


Figure 4.37. Stimulating the students to find the pattern of multiplication of fraction with natural number

Researcher : Can you find any pattern from these two multiplications of fractions after we change the improper fraction into proper fraction?

S3 : I think they have the same denominator.

Researcher : How about the numerator?

S3 : It changes.

Researcher S2 : Now, how you get 91? Pointing out 91 from $\frac{91}{6}$ as a result from $\frac{13}{6} \times 7$.

S1 : Ow, it is from 13 times 7.

Researcher : How about 65? How do you get this? Pointing out 65 from $\frac{65}{4}$ as a result from $\frac{13}{4} \times 5$.

S2 : It is from 13 times 5.

Researcher : Alright, I want to bring S1's idea to put 1 as denominator of the natural number. She wrote as follows:

$$\frac{13}{6} \times \frac{7}{1} = \frac{91}{6}$$

Can you see any relation from the multiplication of fraction after we add 1 as denominator of the natural number?

S3 : I think the denominator is the same, we multiply denominator with denominator.

Researcher : Ok, like this? (see Figure 4.38)

All students : Yes!

Researcher : Oh wow, then we know how to multiply fraction with natural number. We put 1 in the natural number, and then multiply its numerator with numerator, and multiply its denominator with denominator?

S1 Yes! Right, this way much easier.

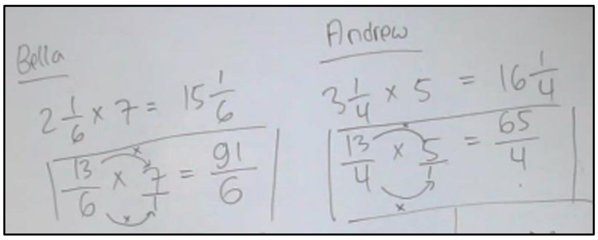


Figure 4.38. Generating formula for multiplication of fraction with natural number

By analyzing the aforementioned discussion, it becomes evident that students can develop a general understanding and eventually formulate a formal multiplication formula for fractions with natural numbers through pattern recognition. This approach aligns with the third tenet of RME, which emphasizes the incorporation of students' contributions into the learning process (Gravemeijer, 1994). RME emphasizes the use of production and construction techniques to assist students in generalizing their knowledge. Building upon the identification of patterns and engagement in concrete experiences, students can progressively abstract their understanding and generalize it to a more formal level, transitioning from a horizontal mathematization to a vertical mathematization process, moving from a concrete to an abstract level. Once students grasp the concept of multiplying fractions with natural numbers within a problem context, they can advance towards formalizing the underlying principles and developing the corresponding mathematical formula.

Following the discussion, the students were assigned Worksheet 2, which required them to solve multiplication problems involving fractions and natural numbers. The subsequent minilesson presented an opportunity for students to reflect on the strategies they employed to solve the marathon investigation problem.

Continuing the discussion, it is notable that S2 and S3 persisted in utilizing the double number line method to solve the problems, as observed in Figures 4.39 and 4.40.

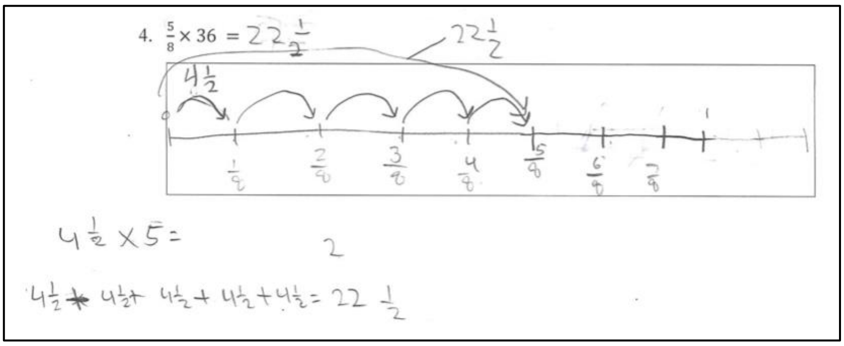


Figure 4.39. Sample of S3's strategy to solve multiplication of fraction with natural number in Worksheet 2 by using double number line model

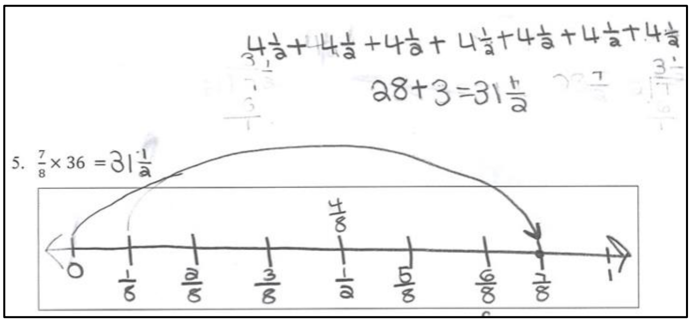


Figure 4.40. Sample of S2's strategy to solve multiplication of fraction with natural number in Worksheet 2 by using double number line model

In contrast, S1 employed a different strategy to solve the multiplication problems involving fractions and natural numbers. S1 utilized the previously discussed formula for multiplying fractions by natural numbers, which involved adding 1 as the denominator of the natural number. This approach is demonstrated in Figure 4.41.

By adding 1 as the denominator in the natural number, S1 effectively converted it into a fraction with 1 as the denominator, allowing for the direct multiplication with the given fraction. This technique simplifies the calculation process and helps students easily compute the product.

The approach employed by S1 aligns with the concept of extending the understanding of multiplication from natural numbers\ to fractions. By treating the natural number as a fraction, the multiplication can be performed in a consistent manner, emphasizing the underlying relationship between natural numbers and fractions.

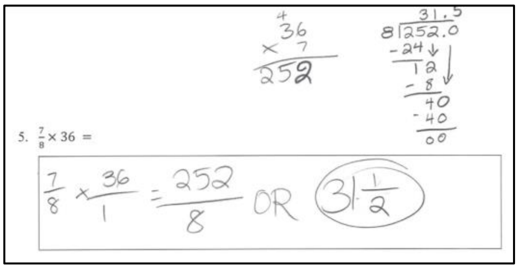


Figure 4.41. Sample of S1’s strategy to solve multiplication of fraction with natural number in Worksheet 2

4.2.2. Ratio Table as Model for Learning Multiplication of Fraction with Natural Number - Training for Next Year’s Marathon Context

The purpose of this activity, outlined in the HLT, was to actively involve students in utilizing landmark fractions and partial products when multiplying a fraction by a natural number. Additionally, the activity aimed to encourage students to explore the concepts of multiplication and division, specifically involving a natural number and a fraction, and understand the relationship between the two operations.

To initiate the activity, the researcher directed the students’ attention to the Training Record table featured in Worksheet 3. The ensuing conversation between the researcher and students is presented below.

Researcher	:	Now we have Andrew and Bella’s friends wanted to exercise for next year’s marathon as did by Andrew and Bella. Let’s have a look at the case of Alex. He ran 4 tracks in 120 minutes, or in other words, in 120 minutes, Alex can complete 4 tracks. Now, in the next column, we have rate which is minutes/circuit. Which means, how long..
S3	:	Divided!
Researcher	:	Again S3?

S3	:	120 divided by 4 equals 30.
Researcher	:	So, how many minutes for 1 track?
S1	:	I know, I know. So, 120 minutes divided by 4 equals 30 minutes and since he went 4 tracks it takes 120 minutes, because 30 times 4 equals to 120. S2 nodded her head supporting S1's idea.
Researcher	:	Thank you, S1. Now, who can explain what is rate?
S2	:	How many minutes it will takes for 1 circuit.
Researcher	:	Yes, right! Thank you, S2

After the initial conversation with the students, the researcher requested them to solve a problem collectively on the whiteboard and then assessed each other's work. The initial task was to determine the rates of Alex, Ethan, and John. All the students were able to solve these problems effortlessly by dividing the time taken by each individual with the circuit length.

Subsequently, they were presented with another set of problems which required them to calculate the time taken by Elizabeth, Benjamin, and Olivia. The students faced no challenges in finding the time for Elizabeth and Benjamin, as they realized that multiplying the rate by the circuit length would yield the answer. However, S3 and S2 had difficulty determining the time taken by Olivia, as it involved multiplying a fraction ($\frac{1}{2}$) with a natural number (8).

To encourage them, the researcher reminded them of the context of the scenario, Running for Fun. S2 stated that they needed to calculate "Half of 18", which equals 9. On the other hand, S3 was still struggling to arrive at the answer. Further examination by the researcher revealed that S3 had employed the formula discussed during the math congress, in which 1 was added to the denominator of 18 (i.e., $\frac{1}{2} \times \frac{18}{1} = \frac{18}{2}$). Initially, S3's response remained as $\frac{18}{2}$. However, with the researcher's guidance, he realized the need to divide 18 by 2 to obtain 9. Once the students completed their calculations, the researcher asked them to review their peers' work and describe the strategies employed to determine the time taken by Olivia if she completed only half of the track in 18 minutes. S1 took this opportunity to explain her strategy and thoughts in the conversation depicted in Figure 4.42.

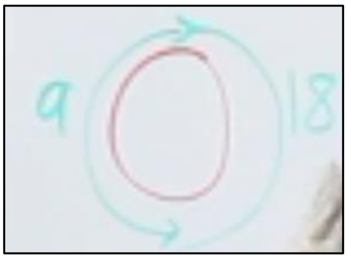


Figure 4.42. S1 explained her strategy to her peers how to find how many minutes Olivia ran

S1	:	Since she completed half of circuit. S1 drew a half of circle. And here is half which is 9, and the whole way is 18.
Researcher	:	Right! Ok, let's discuss again S1's idea. Here we have 18 minutes for 1 track. The researcher drew a circle. But Olivia only completed half of the track. So, from here to here (pointing half circle), how many minutes?
All students	:	9!
Researcher	:	Yes! Actually, you can use the same idea as found by S1 to solve the case of Emma. Emma's rate was 20, but she only completed $\frac{1}{4}$ of the track. Let's draw a circle and mark a quarter of that circle. How many minutes of that quarter?
S2	:	That would be 20 divided by 2?
S3	:	Emm..20 divided by 4?
Researcher	:	Yes! What is the answer?
S3	:	5!
Researcher	:	Good job! Now try to solve the case of Isabella.

In the subsequent task, the students were challenged to determine the time it took Isabella to complete $\frac{3}{4}$of the track, given her rate of 20. S3 adopted the approach suggested by S1, utilizing the circle model to tackle the problem. On the other hand, S2 employed the formula for multiplying fractions with a whole number to arrive at the solution (i.e., $\frac{3}{4} \times 20 = \frac{3}{4} \times \frac{20}{1} = \frac{60}{4} = 15$). Interestingly, S1 incorporated both strategies in her response, as depicted in Figure 4.43.

To solve this problem, S3 utilized the circle model introduced by S1. This model represents the track as a circle, divided it into four equal parts, and shaded three out of these four parts to indicate the completion of $\frac{3}{4}$ of the track. By analyzing the model, S3 was able to determine that Isabella took 15 minutes to complete $\frac{3}{4}$ of the track.

On the other hand, S2 approached the problem by utilizing the formula for multiplying fractions with a whole number. By multiplying $\frac{3}{4}$ with 20, S2 obtained $\frac{60}{4}$. Through simplification, S2 determined that it took Isabella 15 minutes to complete $\frac{3}{4}$ of the track.

Interestingly, S1 combined elements from both strategies to arrive at the solution for Isabella's time. S1 started by using the circle model, similar to S3, to visually represent the track and determine the completion of $\frac{3}{4}$ of it. From there, S1 proceeded to apply the formula for multiplying fractions with the whole number and obtained the same result of 15 minutes.

This demonstrates that the students employed various problem-solving strategies to tackle the task of determining Isabella's time for completing $\frac{3}{4}$ of the track. Each strategy was effective in yielding the correct answer, showcasing the students' understanding and application of mathematical concepts.

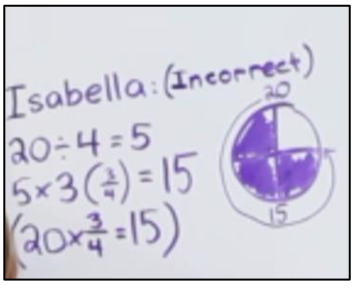


Figure 4.43. S1 strategy to find how many minutes Isabella ran

The students were then presented with another problem to solve involving James, who completed $1\frac{1}{2}$ tracks with a rate of 20. S3 once again utilized the circle model to approach the problem, while S2 employed the formula for multiplying fractions with a whole number. S1, on the other hand, used a combination of both strategies to find the solution.

When faced with the case of Rafa, who completed $2\frac{3}{4}$ tracks with a rate of 30, the students encountered difficulties at the beginning due to the presence of an unfriendly fraction, namely $2\frac{3}{4}$. S2 persevered and continued to use the formula for multiplying fractions with a whole number, as depicted in Figure 4.44.

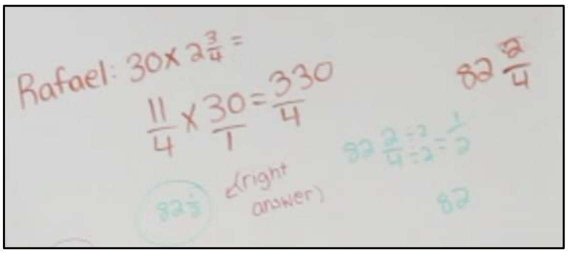


Figure 4.44. S2's strategy to find the minutes of Rafa

S3, on the other hand, decided to represent the track or circuit using the circle model, as shown in detail in Figure 4.45.

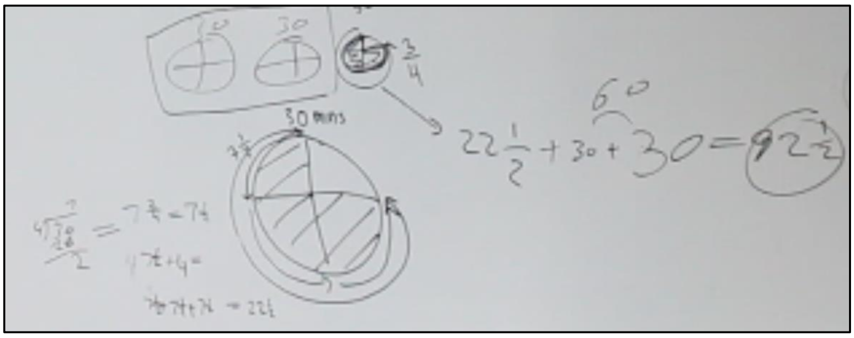


Figure 4.45. S3's strategy to find the minutes of Rafa

S1 chose a different approach by explaining the problem in writing. She began by finding $\frac{3}{4}$ of 30 and then adding the result, which was $22\frac{1}{2}$,, to 60 (representing 2 times the track or 30 multiplied by 2), as illustrated in Figure 4.46.

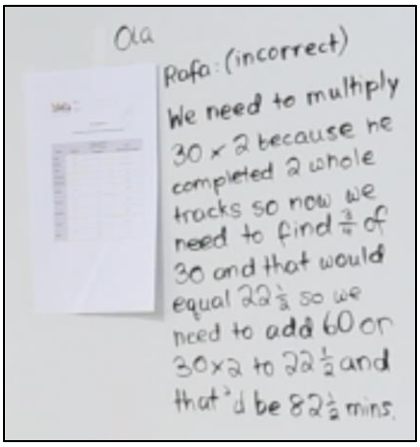


Figure 4.46. S1's strategy to find the minutes of Rafa

Before proceeding with Worksheet 4, Minilesson: "Fractions as Operator", the researcher requested the students' opinions on who the fastest runner among the ten individuals was. The following is a conversation between the students and the researcher as they discussed and debated who held the title of the fastest runner.

S2	:	Alex.
Researcher	:	Can you explain why, S2? S2 was not responding and still thinking.
S1	:	Mmm..I think that was John?
Researcher	:	Alright, I would like to ask you, for the fastest runner, will she or he completed running 1 track with a little of time of a lot of time?
S1	:	I know, John!
Researcher	:	Why John S1?
S1	:	Since he completed 1 circuit in 15 minutes.
Researcher	:	Yes, is it correct, S2?
S2	:	Yes, the fastest is the one with little time to complete 1 track

The conversation mentioned above aligned with the HLT that was conjectured in which the students were able to recognize the patterns and relationships among numbers, leading them to determine that the fastest runner is the one who takes the least amount of time to complete one track.

Following the discussion on the students' problem-solving strategies in the context of Training for Next Year's Marathon, the students were then given a series of problems involving the multiplication of fractions with whole numbers. Both S2 and S3 utilized the formula for multiplying fractions, ($\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$), to solve these problems. However, an interesting strategy was observed from S1, as depicted in Figure 4.47.

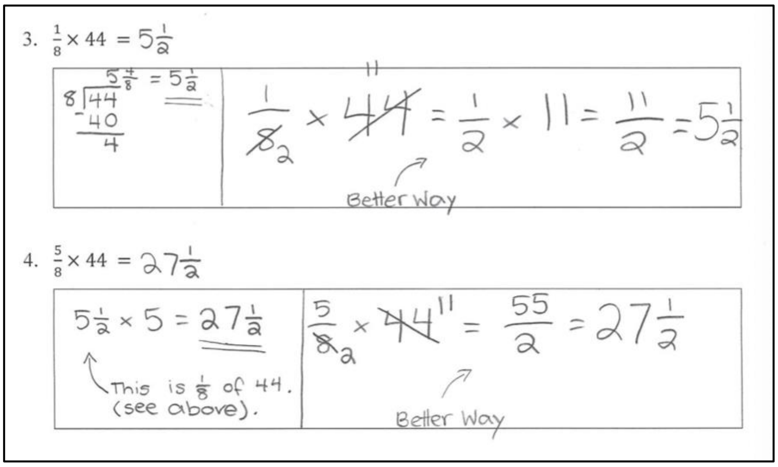


Figure 4.47. S1's strategies to solve question 3 and 4 in Worksheet 4

As seen above, S1 demonstrated a more abstract approach to solving the multiplication of a fraction with a natural number, utilizing partial products and the distributive property. The first strategy she employed involved dividing 44 directly by 8. Drawing from her previous experiences with fraction multiplication problems, she realized that she could divide a natural number by the denominator of the fraction directly. This led her to the solution for question number 3.

S1 then used the answer from question number 3 to solve question 4. Recognizing the relationship between the numbers, she observed that $\frac{1}{8}$ of 44 is equivalent to $5\frac{1}{2}$. With this insight, she was able to determine that to find $\frac{5}{8}$ of 44, she simply needed to multiply $5\frac{1}{2}$ by 5.

To carry out her strategy, S1 utilized the concept of partial products and the distributive property. She decomposed $\frac{5}{8}$ into $\frac{1}{8}\times$ multiplied by 5, resulting in:

$$\frac{5}{8}\times 44=\left(\frac{1}{8}\times 5\right)\times 44=(\frac{1}{8}\times 44)\times 5=5\frac{1}{2}\times 5=27\frac{1}{2}$$

This approach highlights S1's ability to recognize patterns and relationships among numbers, as well as her understanding of mathematical properties such as the distributive property.

In the case of S1, her problem-solving strategy showcased the connection between fractions and real-world scenarios. By using the concept of partial products and the distributive property, S1 was able to break down the multiplication problem in a way that made sense to her. This aligns with the RME tenet of employing realistic contexts, as she used her understanding of fractions and numbers to solve a problem that could potentially arise in practical situations.

Moreover, S1's strategy also reflects another fundamental aspect of RME, namely building on students' existing knowledge and experiences. S1 drew on her previous experiences with fraction multiplication problems, recognizing the pattern of dividing a natural number by the denominator of the fraction. This demonstrates the importance of incorporating students' prior knowledge and experiences into the learning process, as it can significantly enhance their problem-solving abilities and understanding of mathematical concepts.

By utilizing the principles of RME, S1 not only found a solution to the multiplication problem but also developed a deeper understanding of fractions, numbers, and their applications in real-life situations. This highlights the effectiveness of RME in fostering meaningful learning experiences and promoting mathematical proficiency.

These findings are consistent with the research on RME and its impact on students' problem-solving skills and conceptual understanding (Van den Heuvel-Panhuizen, 2003).

Upon further investigation, the researcher asked S1 to elaborate on her second strategy, which involved dividing the natural number directly by the denominator of the fraction. S1 responded by explaining that when considering the fraction $\frac{1}{8}$, she viewed it as representing "1 part out of 8", and since the whole quantity was 44, she needed to divide 44 by 8 to find the value of one part. This interpretation of fractions as parts of a whole allowed S1 to formulate her own calculation method.

Similarly, for question number 4, S1 applied the same strategy. She initially divided 44 by 8 to determine the value of one part, and then proceeded to multiply that result by 5. Finally, she needed to find the value of $\frac{55}{2}$.

S1's approach illustrates her ability to conceptualize fractions in terms of their relationship to a whole and to employ that understanding in her calculations. This aligns with the theories of RME, which emphasize the importance of connecting mathematics to real-life situations and promoting the use of realistic contexts to enhance students' understanding (Van den Heuvel-Panhuizen, 2003; Gravemeijer, 2008).

By perceiving fractions as parts of a whole, S1 demonstrated an intuitive understanding of the concept. This understanding is crucial for developing mathematical proficiency, as it allows students to relate fractions to real-world scenarios and apply their knowledge in problem-solving situations (Lamon, 2007).

S1's strategy also highlights the role of students' prior knowledge and experiences in problem-solving. By drawing on her previous encounters with fraction multiplication problems, S1 was able to identify patterns and apply familiar strategies to solve new problems. This aligns with the principles of RME, which emphasize building on students' existing knowledge to facilitate meaningful learning experiences (Verschaffel et al., 2019).

These findings support the effectiveness of RME in promoting students' conceptual understanding and problem-solving skills, as evidenced by S1's ability to use her understanding of fractions as parts of a whole to devise her own calculation method (Gravemeijer, 2017).

Moreover, S1's strategy can also be analyzed through the lens of vertical mathematization, a concept rooted in didactical phenomenology. According to Lerman (2013), vertical mathematization involves interconnecting diverse mathematical concepts and procedures to establish conceptual coherence. In S1's case, she leveraged her understanding of fraction multiplication to create a coherent strategy for solving problems involving fractions. This showcases the vertical mathematization process, where students integrate multiple mathematical ideas and techniques.

The combination of RME and vertical mathematization in S1's approach demonstrates the interaction between teaching and learning processes. Through the implementation of RME principles, instructional activity created a classroom environment that fosters connections between mathematics and real-life experiences, enabling students like S1 to engage in vertical mathematization. This aligns with the didactical phenomenology approach, which focuses on the relationship between teaching, learning, and the development of mathematical concepts (Vergnaud, 1990).

4.2.3. Array as a Model for Learning Multiplication of Fractions - Exploring Playgrounds and Blacktop Areas Context

During the Exploring Playgrounds and Blacktop Areas context, the array model was introduced as a tool to support students in solving multiplication of fractions problems. The primary goal was for students to develop an array model that could visually represent the relationships between different components, such as the playground and lot, blacktop and playground, and blacktopped-playground and lot.

To ensure a solid foundation for problem-solving, the students were first prompted to discuss their understanding of key concepts related to fractions. They were asked to define what a fraction is, how to represent fractional parts by shading, and the concept of rectangle area. By engaging in this conversation, the students were able to activate their prior knowledge and establish a common understanding of these fundamental mathematical ideas.

Researcher	:	What is the definition of fraction?
S1	:	It's a whole divided into equal parts.
S2	:	Part of a whole.
Researcher	:	Alright, now if I have $\frac{5}{8}$ and I want to represent this fraction in rectangle. What should we do next?
S1	:	We should divided the rectangle into 8 more-or-less equal parts.
Researcher	:	Can I divide it into vertical or horizontal?
S3	:	Yes, you can do either vertical or horizontal.
Researcher	:	Ok, now I am doing the vertical. Anyone wants to help me?
S2	:	Me! S2 shaded the 5 parts of 8 parts from as shown in Figure 4.48.
Researcher	:	Great, thank you, S2. Now, I want to ask you all about the area of rectangle. Does anyone know what is the formula of area of rectangle?
S2	:	We have learned it in grade 4, but I forgot.
S1	:	I know, that would be length times width.
Researcher	:	Great. Please keep this in your mind. Perhaps we will be using this formula to solve the problem that we will read together. Does anyone want to help me to read the problem?

S2 read the Exploring Playground and Blacktop Areas problem.

Researcher	:	So, how many gardens are there?
All students	:	Two, Botany and Gulhane.
Researcher	:	Right, now, are both having the same measurement?
S2	:	Yes, they are the same, because it says here "two of them are measures 50 meters by 100 meters.
Researcher	:	Exactly, now please re-read the following sentence. "One lot, the residents agreed that $\frac{3}{4}$ of the lot will be devoted to a playground for children and then $\frac{2}{5}$ of that playground will be covered by blacktop, so children can play basketball.". Can you please explain to me whether the blacktop will be part of the playground or not?

S1 : Yea, I think so. Blacktop is where to play basketball and it is still inside the playground.

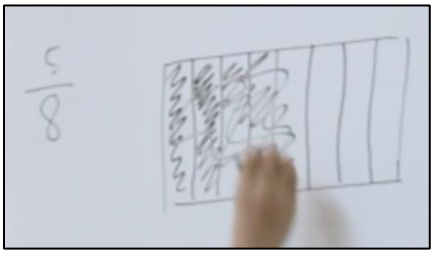


Figure 4.48. S2 shaded the 5 parts of 8 parts to show $\frac{5}{8}$

Upon reviewing the students' worksheets, it was discovered that all students attempted to depict Botany and Gulhane gardens as rectangular shapes. Extending from their prior understanding of fractional parts, the students endeavored to determine the fraction of blacktop in relation to the playground area. Specifically, for the Botany garden, they aimed to find $\frac{2}{5}$ of the blacktop in $\frac{3}{4}$ of the playground, while for Gulhane garden, they sought to ascertain $\frac{3}{4}$ of the blacktop in $\frac{2}{5}$ of the playground. These efforts align with the conjecture of students' thinking in the HLT, which suggests that the students utilized the array model as a means to model fractions.

Subsequently, after the students completed documenting their strategies on the whiteboard, S1 initiated a discussion by expressing her curiosity regarding whether one lot had a greater blacktop area than the other. To address this question, she recognized the need to determine the exact areas of the blacktop in both Botany and Gulhane gardens. This consideration is evidenced by her calculation of the blacktopped areas for both gardens, as depicted in Figure 4.49.

The researcher affirmed that S1's strategy was accurate and could serve as evidence to convince the residents of the two neighborhoods that her conclusion was correct.

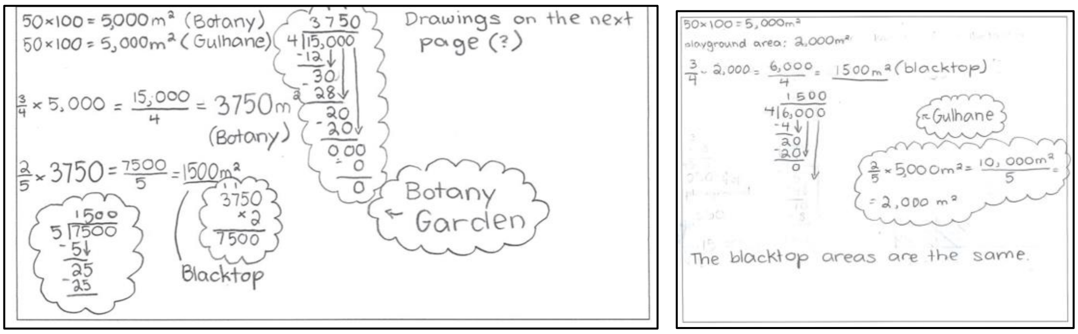


Figure 4.49. S1's strategy to find which garden has more blacktop

To support the findings and provide a broader understanding of the significance of the array model in fraction representation, it is necessary to refer to relevant literature. One such framework is the RME approach, which emphasizes the relevance of contextualization in mathematics instruction (Gravemeijer, 2017). The incorporation of real-life contexts, such as gardens and playgrounds, allows students to grasp the underlying concepts of fractions and apply their understanding to solve authentic problems.

Moreover, the student's ability to initiate discussions and pose relevant questions within the mathematical discourse reflects their critical thinking skills. This aligns with the aim of cultivating mathematical reasoning and communication abilities in students (NCTM, 2000). By promoting student-led discussions and incorporating their inquiries into the learning process, there is a notable improvement in their active engagement and personal investment in their learning experience. Students' application of the array model in representing fractions within the context of Botany and Gulhane gardens underscores their comprehension and effective utilization of mathematical concepts. By promoting meaningful discussions and encouraging students to generate relevant questions, their participation and depth of understanding are greatly enriched.

Before we further discuss S1's strategy, S2's and S3's drawings of $\frac{2}{5}$ blacktop of $\frac{3}{4}$ playground (for Botany garden) and $\frac{3}{4}$ blacktop of $\frac{2}{5}$ playground (for Gulhane garden) were brought into a discussion since they both used drawings to find which garden has more blacktop. Their strategies are shown in Figures 4.50 and 4.51 below.

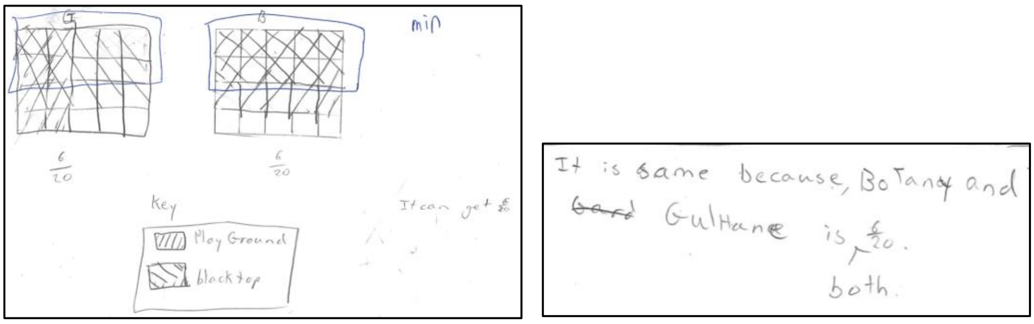


Figure 4.50. S3's strategy to find which garden has more blacktop

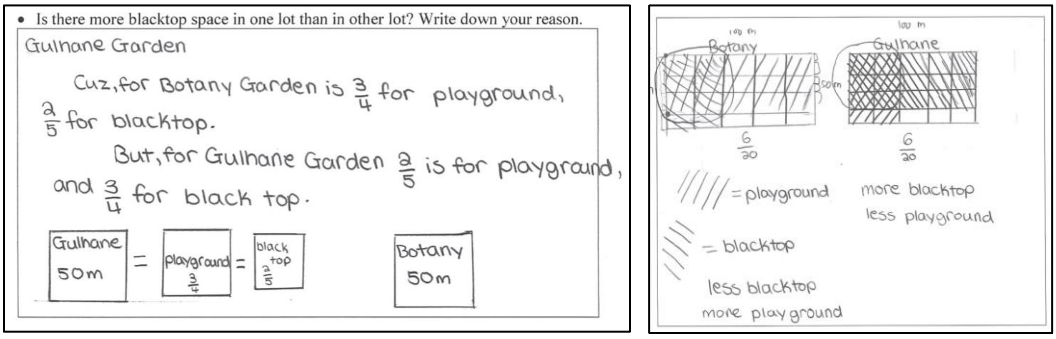


Figure 4.51. S2's strategy to find which garden has more blacktop

The following conversation sheds light on the diverse approaches utilized by the students, allowing for a deeper understanding of their findings.

Researcher	:	S2, could you please explain your drawing?
S2	:	Ok, first for Botany, it says the residents agreed that $\frac{3}{4}$ of the lot will be devoted to a playground and then $\frac{2}{5}$ of that playground will be covered by blacktop. So, I first divided the rectangle into 4 like this. S2 gave a sign that she divided horizontally. And then, I shaded 3 parts out of 4, and that's the playground.

After that, to find the blacktop, I divided the rectangle into 5 parts like this. S2 gave a sign that she divided the rectangle vertically. I shaded 2 parts from 5 parts and found the parts that shaded two times. In fraction, I can write it as $\frac{6}{20}$.

Researcher	:	Very well explained, S2. Good job. Now I would like to ask to all of you, is blacktop part of the playground or outside the playground?
S1	:	Blacktop is part of the playground and also part of the lot.
Researcher	:	Yes! The lot is the whole and we will still consider the lot as a whole. For Gulhane, S3, do you want to explain your drawing?
S3	:	It is the same as Botany, but first I need to find $\frac{2}{5}$ first, and then $\frac{3}{4}$ and the answer is still $\frac{6}{20}$ for the blacktop in Gulhane.
Researcher	:	Great, S3. Anyway, S3 and S2, how can you get $\frac{6}{20}$?
S2	:	Like it shaded 2 times.
Researcher	:	Alright, shaded 2 times. How many blocks are shaded 2 times?
S3	:	6
Researcher	:	And what is the fraction for the parts that shaded two times?
S2 and S3	:	$\frac{6}{20}$
Researcher	:	Yes, do not forget, we include the parts of the lot as it is the whole. So, we counted 20 parts as the whole, and 6 as the parts of blacktop and that the fraction becomes $\frac{6}{20}$. Well, S3, could you please explain your conclusion whether there is more blacktop in Botany than in Gulhane?
S3	:	No, they are the same. Because, the fractions is $\frac{6}{20}$ for both Gulhane and Botany gardens.
S2	:	Why would it be the same because the blacktop is more in Gulhane than in Botany?
S1	:	I think that's because we know that for Botany, $\frac{3}{4}$ for playground and $\frac{2}{5}$ for blacktop, and for Gulhane it switched. They are the same area, but they switched.

Based on the above conversation, it is evident that S2 successfully used drawings to determine the proportions of blacktop in both the Botany and Gulhane gardens. However, S2 found that Gulhane garden has more blacktop and explained that "for

Botany Garden is $\frac{3}{4}$ for playground, $\frac{2}{5}$ for blacktop. But for Gulhane garden, $\frac{2}{5}$ is for playground, and $\frac{3}{4}$ for blacktop".

During the discussion, the researcher encouraged S2 to remember the keyword "of" in order to clarify the concept of multiplication. Together, they realized that the fraction of the blacktopped area in the Botany garden could be obtained by multiplying $\frac{2}{5}$ by $\frac{3}{4}$ ($\frac{2}{5} \times \frac{3}{4}$), representing $\frac{2}{5}$ of $\frac{3}{4}$. Similarly, the fraction of the blacktopped area in the Gulhane garden could be obtained by multiplying $\frac{3}{4}$ by $\frac{2}{5}$ ($\frac{3}{4} \times \frac{2}{5}$), representing $\frac{3}{4}$ of $\frac{2}{5}$.

Furthermore, the researcher introduced the concept of the commutative property of fractions by using the word "switched" based on S1's earlier idea. This led the students to discover that the result of $\frac{2}{5}$ multiplied by $\frac{3}{4}$ is equal to $\frac{3}{4}$ of $\frac{2}{5}$, which is $\frac{6}{20}$.

According to the RME approach, students progress through different levels of mathematization, from horizontal to vertical (Treffers, 1987, 1991a; Freudenthal, 1991; Streefland, 1991; Gravemeijer, 1994, 2008). In horizontal mathematization, students focus on solving mathematical problems in a concrete and context-specific manner. They engage in manipulations and calculations without a deep understanding of underlying mathematical principles. As they advance to vertical mathematization, students begin to recognize and generalize mathematical structures and concepts, creating connections between different problem situations.

In the given scenario, S1 and S2's discussions reflect a movement from horizontal to vertical mathematization. Initially, they approached the problem by using drawings to represent the gardens and determine the proportions of blacktop. This exemplifies a concrete and context-specific approach. However, as the conversation progresses, the researcher encourages S2 to think more abstractly and consider the concept of multiplication using keywords like "of". This shift in thinking demonstrates their transition to vertical mathematization.

Furthermore, during this process, the students made a significant discovery – they found that the commutative property observed in natural numbers also holds true for fractions. This discovery signifies a higher level of generalization and abstract reasoning. The commutative property states that the order of multiplication does not affect the result. In this case, the students realized that multiplying $\frac{2}{5}$ by $\frac{3}{4}$ yields the same result as multiplying $\frac{3}{4}$ by $\frac{2}{5}$. They express this finding as $\frac{2}{5} \times \frac{3}{4} = \frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$, which confirms the commutative property in regards to fractions.

This insight aligns with the principles of RME, which emphasize the importance of connecting mathematical concepts to real-world contexts and encouraging students to discover and generalize mathematical properties. As they progress through the levels of mathematization, students developed a deeper understanding of mathematical principles and their application in various situations.

The commutative property was then used by the students to solve problems in "Minilesson: Multiplication of Fractions" in Worksheet 6.

S1 tried to bring into discussion about her strategy in which she tried to find the blacktop areas of Botany and Gulhane gardens. She used the dimensions of the lots which is 50 meters × 100 meters, given in the context. As seen in Figure 4.51, she first tried to find the area of the lot/garden, namely $50 \times 100 = 5{,}000\ m^2$. For Botany, she first tried to find $\frac{3}{4} \times 5{,}000 = \frac{15{,}000}{4} = 3{,}750\ m^2$ and then calculated $\frac{2}{5} \times 3{,}750 = \frac{7{,}500}{5} = 1{,}500\ m^2$. She found that the blacktopped area in Botany garden is $1{,}500\ m^2$. The same strategy was used by S1 to find the blacktop area of Gulhane garden. She first calculated the playground area from $\frac{2}{5} \times 5{,}000 = \frac{10{,}000}{5} = 2{,}000\ m^2$ and then calculated $\frac{3}{4} \times 2{,}000 = \frac{6{,}000}{4} = 1{,}500\ m^2$. She found that the blacktop areas in both gardens are the same, namely $1{,}500\ m^2$.

While S1's solution was correct, she encountered difficulty when asked to illustrate the lots, playground, and blacktop using models. The fractional parts posed a challenge for her, as she cut both the fifths and fourths vertically (Figure 4.52). This aligns with the hypothesis in the HLT that states students may struggle to determine the fractional part when cutting vertically or horizontally. Consequently, instead of relying on the model, S1 resorted to using the dimensions of the lot, 50 meters by 100 meters, to find the blacktop area (Figure 4.52).

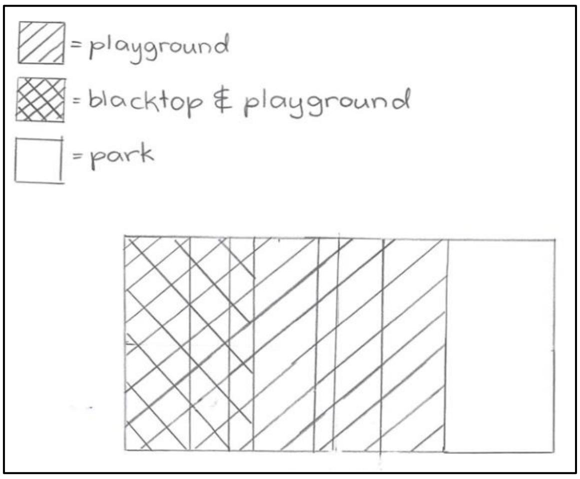


Figure 4.52. S1's drawing for the case of Botany garden

This observation highlights the importance of the RME approach, which emphasizes students' construction of mathematical knowledge through contextual problem-solving and the recognition of mathematical structures. The implementation of manipulatives and drawings serves as a valuable tool for students to visualize the problem and develop a deeper understanding of fractions and their operations (Schoenfeld, 2016).

4.2.4. Array as a Model for Learning Multiplication of Fractions - Comparing the Cost of Blacktopping Context

In this activity, the students were tasked with employing the array model to solve multiplication problems involving fractions. Additionally, they were encouraged to delve deeper into their exploration of the commutative property of multiplying fractions, extending their understanding to percentages and decimals as well.

Prior to commencing the problem-solving task, the researcher inquired whether the students had any prior knowledge of the concept of percentages. However, the students responded that they had not yet been introduced to this concept within their classroom setting. Taking this into account, the researcher took the initiative to establish a connection between the concept of percentages and its significance in daily life. To facilitate this understanding, an example of a "discount" was introduced, as detailed in the following excerpt.

Researcher	:	Actually, in daily life you can find percentage. For example, when we go shopping, there is a percentage, 50% discount.
S3	:	Sale?
Researcher	:	Yes, now if you get 50% discount, how much you'll need to pay?
S1	:	Half of it!
Researcher	:	How can you come up with that idea, S1?
S1	:	Because 50% is half of the price.
Researcher	:	Well, actually for percentage, there is a relation with fraction.
S2	:	Maybe part of something?
Researcher	:	Yes, part of something, what is this something?
S2	:	The whole?
Researcher	:	Yes, what is the whole?
S1	:	100?
Researcher	:	Yes, in the US, cent means a hundredth part of the unit of the money system. 100 cents equal to 1 dollar. Ok, now we have 10%. We can write it in fraction as $\frac{10}{100}$. Can you simplify this?
S2	:	One tenth ($\frac{1}{10}$)
Researcher	:	How about 25%?
S1	:	One fourth ($\frac{1}{4}$)
Researcher	:	How you come up with $\frac{1}{4}$, S1?
S1	:	Because 25% is $\frac{25}{100}$ and after you simplify it, you get $\frac{1}{4}$

Through the aforementioned conversation, it became evident that the students were able to grasp the concept of percentages after being presented with several examples. They understood that a percentage represents a fraction out of a hundred, indicating a proportion or relative quantity. During the discussion, the students also observed the simplification aspect of percentages, realizing that dividing both the numerator and denominator by the same number would maintain the equivalent fraction.

Before delving into the primary topic of the day's activity, which revolved around comparing the cost of blacktopping, the students were prompted to recall the previous context of exploring playground and blacktop areas. To facilitate this recollection, the researcher called upon one of the students, S3, to visually represent the case of a blacktopped area in the Botany garden. The student's drawing is depicted in Figure 4.53 below.

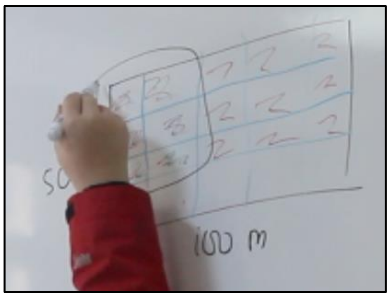


Figure 4.53. S3 drew the array model of blacktopped area

Subsequently, the researcher urged the students to determine the area of the blacktop. Based on their previous knowledge, the students recognized that the fraction representing the blacktopped area was $\frac{6}{20}$, as illustrated in Figure 4.53. Their task now was to determine $\frac{6}{20}$ of the garden area, which was 50 meters by 100 meters, or, in other words, $\frac{6}{20}$of 5,000 m2. To solve this, the students employed the multiplication algorithm for fractions with a natural number, as follows:

$$\frac{6}{20} \text{ of } 5{,}000\ m^2 = \frac{6}{20} \times 5{,}000 = \frac{6}{20} \times \frac{5{,}000}{1} = \frac{30{,}000}{20} = 1{,}500\ m^2$$

Once the students found the area of the blacktop, which was equivalent in both Botany and Gulhane gardens, they confronted the main problem, which was to compare the cost of blacktopping in the two gardens based on the prices and discounts provided by the contractor.

In the given context, it was stated that “the cost of blacktopping in the first lot, Botany garden, is \$9 per square meter, but the contractor will offer to do it at 80% of that price”. As for the second lot, “the contractor charges \$8 per square meter, but the contractor will offer to do it at 90% of that price”. The students were tasked with determining whether the cost of blacktopping would be higher in one of the parks compared to the other. After being given sufficient time to solve the problem, it was discovered that S2’s and S3’s strategies were the same, as depicted in Figures 4.54 and 4.55.

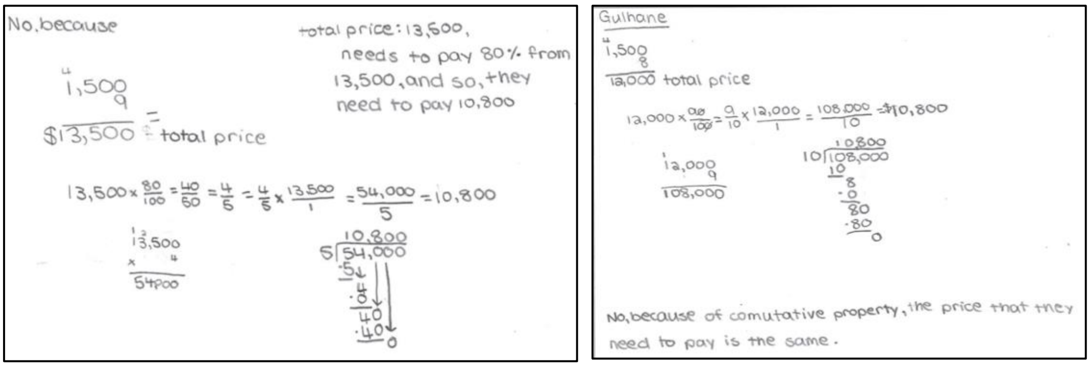


Figure 4.54. S2’s strategy to find the costs of blacktopping in Botany and Gulhane gardens

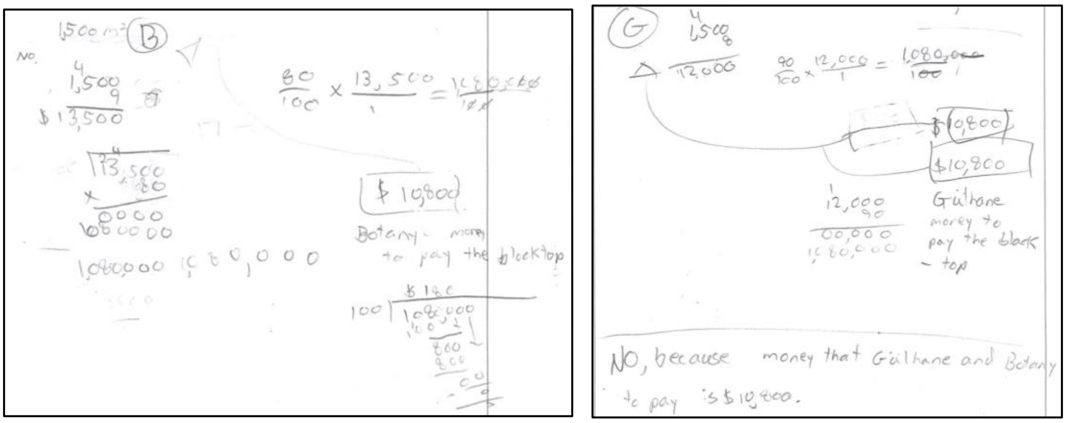


Figure 4.55. S3’s strategy to find the costs of blacktopping in Botany and Gulhane gardens

Based on the strategies employed by S2 and S3, it is evident that they calculated the cost of blacktopping using the determined area of 1,500 m2. They multiplied the area by the respective prices for Botany and Gulhane gardens and then applied the agreed-upon percentage discounts offered by the contractor. For instance, in the case of Botany garden, they multiplied \$9 by 1,500 to calculate the total cost of blacktopping, which amounted to \$13,500. They then determined 80% of \$13,500 by computing 80% of \$13,500 (i.e., $\frac{80}{100} \times 13{,}500 = \frac{80}{100} \times \frac{13{,}500}{1} = \frac{4}{5} \times \frac{13{,}500}{1} = \frac{54{,}000}{5} = \$10{,}800$). Likewise, by using \$8 and 90% of \$12,000, they determined the cost of blacktopping in Gulhane (i.e., $\frac{90}{100} \times 12{,}000 = \frac{90}{100} \times \frac{12{,}000}{1} = \frac{9}{10} \times \frac{12{,}000}{1} = \frac{108{,}000}{10} = \$10{,}800$). After computing the costs for both gardens, the students concluded that the community would have to pay the same amount for blacktopping in both Botany and Gulhane gardens.

S2's and S3's strategies were in line with the conjecture of student thinking in the HLT, namely "the students may multiply the area with the price per meters and then found out the discount".

In contrast to S2 and S3, S1 took a different approach by attempting to determine the price for blacktopping per square meter after applying the respective discounts (i.e., 80% of \$9 for Botany and 90% of \$8 for Gulhane).

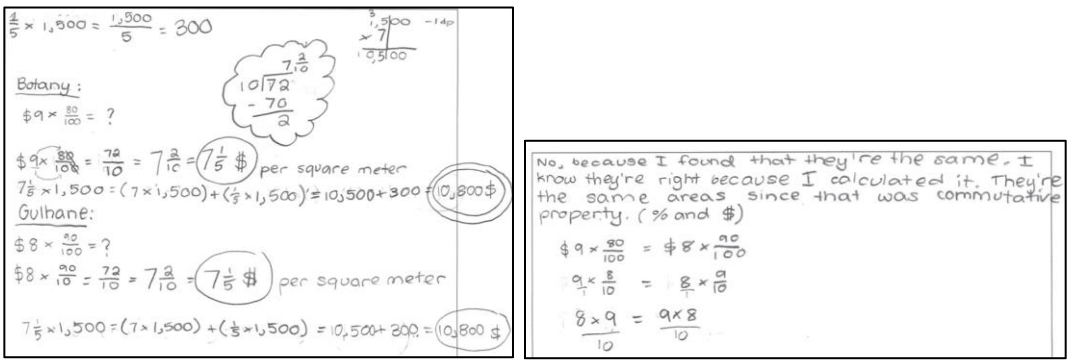


Figure 4.56. S1's strategy to find the costs of blacktopping in Botany and Gulhane gardens

As depicted in Figure 4.56, S1 calculated that the cost of blacktopping per square meter in both Botany and Gulhane gardens was identical at $\$7\frac{1}{5}$. To find the total cost, she simply multiplied this price by the blacktopping area of $1{,}500\ m^2$. To perform this calculation, S1 utilized the distributive property since she was dealing with the fraction $7\frac{1}{5}$. She decomposed the fraction multiplied by a whole number: $7\frac{1}{5} \times 1{,}500 = (7 \times 1{,}500) + \left(\frac{1}{5} \times 1{,}500\right) = 10{,}500 + 300 = \$10{,}800$. Her strategy aligns with the conjectured of students' thinking, which suggests that the students might utilize the expressions 80%×\$9 and 90%×\$8 to compare the costs of blacktopping.

Additionally, in her conclusion, S1 further observed that the equivalence underlying the commutative property holds true:

$$\$9 \times \frac{80}{100} = \$8 \times \frac{90}{100}$$

$$\frac{9}{1} \times \frac{8}{10} = \frac{8}{1} \times \frac{9}{10}$$

$$\frac{9 \times 8}{10} = \frac{8 \times 9}{10}$$

The approach taken by S1 in solving the problem of determining the cost of blacktopping per square meter after discounts aligns with the principles of RME. RME emphasizes the use of real-world contexts and situations to engage students in meaningful mathematical tasks and problem-solving (Gravemeijer, 2004). By considering the discounts and applying them to the given prices, S1 is mathematizing the real-world situation, transforming it into a mathematical problem that can be solved using mathematical concepts and operations.

Furthermore, S1's strategic use of the distributive property to solve the multiplication involving the fraction $7\frac{1}{5}$ demonstrates both vertical and horizontal mathematization. Vertical mathematization involves the progression of mathematical knowledge and skills within a specific mathematical domain, in this case, fractions and multiplication. S1 applies her understanding of multiplying fractions with natural numbers to decompose the expression and simplify the calculation.

Horizontal mathematization, on the other hand, involves the ability to transfer mathematical knowledge and skills across different mathematical domains. S1's use of the distributive property to solve a problem that involves percentages, fractions, and multiplication showcases her capacity to apply a mathematical concept learned in one context to solve a problem in a different context (Wijaya, 2014).

S1's thought process, as reflected in her strategy and conclusion, can also be analyzed from the perspective of Didactical Phenomenology. Didactical phenomenology focuses on understanding how students perceive and interpret mathematical situations, and how their thinking is influenced by the didactical contract established by the teacher (Brousseau, 1997). In this case, S1's thinking follows the conjecture of students' thinking suggested in the study, which is based on their understanding of percentages and their ability to compare and reason about costs.

Therefore, S1's approach to solving the problem reflects the integration of RME principles, the process of mathematizing, the concepts of vertical and horizontal mathematization, and the insights provided by didactical phenomenology. These perspectives collectively highlight the importance of connecting mathematical concepts to real-world situations, reasoning across multiple mathematical domains, and understanding the students' thought processes in mathematical problem-solving.

Upon completion of activity 7, the students were given Worksheet 8, which involved a Minilesson titled "Interchanging Numerators". The objective of this lesson, facilitated by the researcher, was to explore the concept of interchanging numerators (or denominators) to determine the product of two fractions. To illustrate this, question number 3 was chosen as an example, where the students were tasked with solving $\frac{3}{5}$ multiplied by $\frac{2}{3}$.

Initially, the students utilized the multiplication of fractions formula to solve the problem, resulting in the expression $\frac{3}{5}$ multiplied by $\frac{2}{3}$. They then proceeded to simplify the expression further, obtaining $\frac{6}{15} = \frac{2}{5}$.

At this point, the researcher encouraged the students to analyze the relationships between the numerators and denominators. Upon careful examination, the students observed that the numerator 3 in the first fraction could be canceled out with the denominator 3 in the second fraction.

The researcher explained to the students that this process of cancellation involved interchanging the numerators, as demonstrated in Figure 4.57. By interchanging the numerators, the fraction became simplified, resulting in the final answer.

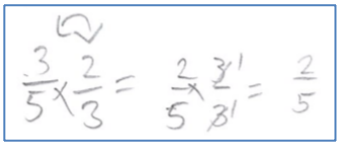


Figure 4.57. Using interchanging numerators to derive the product of two fractions

During the discussion on interchanging numerators within the Minilesson presented in Worksheet 8, the students were given an opportunity to enhance their understanding of fraction multiplication and simplification. Through the researcher's guidance and encouragement to explore the relationships between numerator and denominator, the students gained a deeper comprehension of this fundamental concept.

Through the discussion on interchanging numerators within the Minilesson on Worksheet 8, students had the opportunity to develop their number sense and enhance their understanding of how numbers are related to one another.

When analyzing the problem $\frac{3}{5}$ multiplied by $\frac{2}{3}$, the students recognized that the numerator of the first fraction (3) could be canceled out with the denominator of the second fraction (3). This cancellation involved interchanging the numerators, which resulted in the simplified expression of $\frac{2}{5}$. This observation enabled the students to simplify the fraction further and obtain the final result.

The ability to cancel out numerators and denominators by interchanging them demonstrates the students' growing number sense. Number sense refers to a deep understanding of numbers and their relationships, allowing individuals to recognize patterns, make connections, and manipulate numbers effectively in various mathematical contexts (Kilpatrick, Swafford, & Findell, 2001).

Developing number sense is a crucial aspect of mathematics education as it enables students to develop a deep understanding of mathematical operations and concepts. With strong number sense, students can recognize relationships, spot patterns, and apply their knowledge to solve complex problems with precision and efficiency (NCTM, 2014).

By engaging in the discussion on interchanging numerators, students were able to strengthen their number sense and expand their mathematical reasoning abilities. This foundational knowledge will benefit them as they continue to explore more complex mathematical concepts and solve problems in various mathematical contexts.

The students then continued to solve the problems in Worksheet 8 and used the idea of interchanging numerators as seen in Figure 4.58 below as a sample.

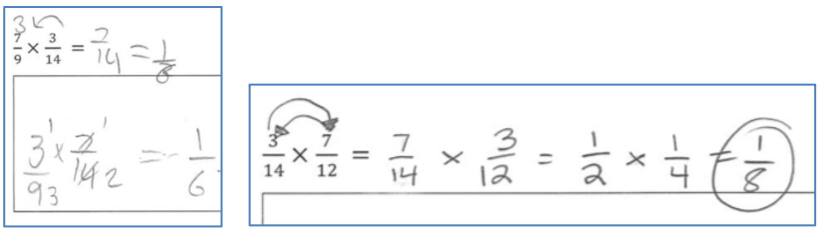


Figure 4.58. Students' works in multiplication of fractions by using the idea of interchanging numerators

In the context of interchanging numerators, students could use their knowledge of fractions to simplify complex calculations and understood how different numbers interact with each other. By recognizing relationships between the numerators and denominators of different fractions, students could manipulate and rearrange these numbers to simplify calculations and solve problems (Kilpatrick, Swafford, & Findell, 2001). Interchanging numerators is a strategy that can be used to simplify fraction multiplication (Cai & Lester, 2010). Interchanging numerators is a useful strategy because it simplifies the multiplication of fractions by allowing students to work with natural numbers instead of fractions. This strategy can be especially helpful when working with complex fractions that have multiple terms (Cai & Lester, 2010). However, it is important for students to understand the concept of interchanging numerators and to be able to explain why it works to simplify fractions (Simon & Tzur, 2004). In this study, this example highlighted the importance of number sense in mathematics education and how it can be developed through activities such as interchanging numerators. By understanding how numbers relate to each other, students can better understand mathematical operations and solve problems with greater ease and accuracy.

In addition, through the discussion on interchanging numerators within the Minilesson on Worksheet 8, the students were engaged in a process that aligns with the principles of RME. According to RME, mathematics learning should be situated in meaningful contexts and allow students to construct their own mathematical understanding (Freudenthal, 1991). The task of interchanging numerators provided an opportunity for students to make connections between different fractions and develop a deeper understanding of fraction multiplication.

The concept of interchanging numerators also aligns with the idea of horizontal and vertical mathematization, which are important principles in mathematical thinking (Gravemeijer, 1999). Horizontal mathematization refers to the process of identifying mathematical relationships within a particular mathematical topic. In this case, the students recognized the relationship between the numerator of one fraction and the denominator of another, leading them to interchange the numerators. This horizontal mathematization allowed them to simplify the expression and obtain the final answer.

Vertical mathematization, on the other hand, refers to the process of identifying mathematical relationships that cut across different mathematical topics (Gravemeijer, 1999). Through the exploration of interchanging numerators, the students were able to see the connections between fraction multiplication and fraction simplification. This vertical mathematization enabled them to transfer their knowledge and apply it in other mathematical contexts.

By engaging in horizontal and vertical mathematization, the students not only developed their understanding of fraction multiplication and simplification but also honed their ability to recognize patterns, make connections, and apply their mathematical knowledge in various situations. This approach to learning mathematics promotes a deeper and more robust understanding of mathematical concepts, laying the foundation for future mathematical reasoning and problem-solving.

CHAPTER 5

DISCUSSION AND CONCLUSION

In this study, a Hypothetical Learning Trajectory (HLT) for learning fraction multiplication and its learning sequence designed using the RME approach was developed, tested, and revised to contribute to a potential local instruction theory for students' learning of fraction multiplication. This chapter presents a synopsis of the findings, including answers to the research questions, namely (ii) how do mathematizing processes facilitate fifth-grade students' learning of multiplication of fractions using Realistic Mathematics Education-based instructional activities?, and (ii) what obstacles do fifth-grade students encounter when learning multiplication of fractions using Realistic Mathematics Education-based instructional activities?.

5.1. Mathematizing Processes throughout the Multiplication of Fractions Activities Designed within the Context of RME

One of the key tenets of RME is the suggestion that mathematics is a human activity (Freudenthal, 1973, 1991). According to Freudenthal, mathematics is not a collection of knowledge, but it is the process of solving and discovering problems, as "there is no mathematics without mathematizing (Freudenthal 1973, p. 134)."

This view of mathematics as a process significantly influenced the development of mathematics education as a discipline. To be more precise, it affected both the goals and the techniques of teaching mathematics. According to Freudenthal (1973, 1991), the greatest way to acquire mathematics is through doing, and mathematizing is the primary goal of mathematics education.

Modeling, symbolizing, generalizing, formalizing, and abstracting are all part of the mathematizing process and they are key aspects of mathematizing in RME (Streefland, 1991). Mathematizing processes are fundamental to supporting students in building their understanding of mathematical concepts, such as multiplication of fractions. Utilizing multiple models, such as number lines, ratio tables, and area/arrays, could facilitate students in developing their conceptual understanding of fraction multiplication.

This study was a design-based research project that consisted of two phases: the pilot experiment and the teaching experiment. Both experiments focused on exploring the development of fifth-grade students' understanding of multiplication of fractions and investigating the effective methods of supporting and organizing this development. The study drew on the principles of design-based research, which emphasizes the iterative process of designing, implementing, and refining instructional interventions.

The goal of the pilot experiment was to test the initial version of the HLT and identify the obstacles and misconceptions that students had regarding multiplication of fractions. The results of this experiment informed the design of the revised HLT and instructional sequence that were used in the teaching experiment.

The teaching experiment focused on implementing the revised HLT and instructional sequence in a fifth-grade classroom and recording the interactions between the instructor (in this case was the researcher), students, and the instructional materials. By doing so, the study aimed to offer a detailed picture of the learning trajectory for multiplication of fractions as it was actualized in the classroom.

To develop and revise the HLT and instructional sequence for multiplication of fractions, this study drew on prior research of students' learning in this area. The work of Hellman and Fosnot (2007) and Holt et al. (2003) served as the foundation for the HLT and instructional sequence developed in this study. The instructional sequence aimed to help fifth-graders build an understanding of multiplication of fractions through engaging with realistic contexts and tasks.

The challenges in learning mathematics are often associated with the gap between informal knowledge based on everyday experiences and the formal mathematics taught through instruction. However, in this study, informal and formal knowledge were not treated as separate entities. Students were encouraged to reinvent formal mathematical knowledge regarding multiplication of fractions by drawing on their informal and intuitive knowledge, as reported by previous researchers studying young children's understanding and development of this concept. This informal and intuitive knowledge is often referred to as qualitative or personal knowledge by researchers such as Lesh & Doerr (2000) and Streefland (1991).

By building on previous research and embracing the informal and intuitive knowledge of students, this study aimed to develop an effective HLT and instructional sequence for teaching multiplication of fractions. The study's approach highlighted the importance of considering students' existing understandings and experiences when designing instructional materials and developing educational interventions, in order to better support their learning and promote a deeper understanding of mathematical concepts.

8 learning activities with 4 contexts including Running for Fun, Training for Next Year's Marathon, Exploring Playground and Blacktop Areas, and Comparing the Cost of Blacktopping were documented. The analysis method proposed by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008) was used to document the mathematical practices in the classroom. The documentation revealed that learning activities had significant potential in supporting the classroom community's fractions multiplication in progressively sophisticated ways.

The documentation of classroom mathematical practices provided valuable insights into the effectiveness of the instructional sequence in promoting fractions multiplication. The learning activities were found to be well-designed, engaging and formulated to promote a deep understanding of the topic among students. The four contexts within which these activities were carried out further reinforce the effectiveness of the sequence in real-world scenarios and adding to the retention of the knowledge by the students.

In this research, four different contexts have been utilized to examine the mathematizing processes within each of them. These contexts were Running for Fun, Training for Next Year's Marathon, Exploring Playground and Blacktop Areas, and Comparing the Cost of Blacktopping. In the upcoming sections, we will delve deeper into each of these contexts, exploring how mathematizing processes gradually developed from modeling to abstraction level of learning multiplication of fractions using activities designed within the context of RME and what were the obstacles that the students encountered during the learning processes.

5.1.1. Mathematizing Processes of Running for Fun Context

Modelling Level

In this study, the researcher explored how the context of Running for Fun provided an opportunity for students to experiment with using a number line as a model to represent the marathon route. This idea was sparked by one student's epiphany that the running route could be visualized as a stretched line, prompting further discussions among the students. Consequently, the students began to consider the number line as a potential representation for measurement contexts.

The utilization of a number line as a model for measurement context aligns with the findings of previous studies. Hilton and Bell (2016) demonstrated in their research the effectiveness of the number line in conveying measuring contexts, such as length and distance. Similarly, Rasmussen and Stephan (2011) also highlighted the utility of the number line as a tool for representing measurement concepts.

By integrating the concept of a number line into the Running for Fun context, the study aimed to enhance students' understanding of how measurements can be represented and interpreted. This approach not only provided a visual representation for the marathon route, but it also allowed students to engage with the mathematical concept of measurement in a tangible and relatable manner.

In addition, the measurement context in in this study has shown to lead students to the number line model, ultimately facilitating their understanding of fractions as relations, equivalent fractions, and the application of properties similar to those seen in natural numbers. Research has consistently shown the positive impact of using number lines as a visual representation in teaching fractions multiplication.

In a study conducted by Mousley and Lowrie (2015), the effectiveness of using number lines to teach fractions multiplication was explored with primary school students. The results indicated that the number line model greatly enhanced students' conceptual understanding of fractions multiplication. Students found it easier to visualize the multiplication process and comprehend the concepts of fractions multiplication when using number lines (Mousley & Lowrie, 2015). This aligns with the findings of Kar (2011), who also reported that the use of number lines enabled students to grasp fractions concepts more easily and effectively.

The utilization of visual representations, such as number lines, as a teaching aid for fractions multiplication has been supported by multiple studies. Gawronski and Wickstrom (2011) and Philipp and Schappelle (2012) both suggest that visual representations are highly effective in promoting mathematical understanding. The visual nature of number lines not only contributes to students' comprehension of fractions multiplication but also helps them appreciate the abstract nature of fractions and mathematical operations in general (Mousley & Lowrie, 2015).

The incorporation of number lines as a visual aid in teaching fractions multiplication has proven to be beneficial for students in understanding the concepts and processes involved. By providing a visual representation, number lines allow students to make connections between the physical measurements and the abstract mathematical representation of fractions. This approach supports their conceptual understanding and facilitates their ability to apply fraction multiplication skills in various contexts.

Symbolizing Level

In the next phase of mathematizing, known as symbolizing, the students in the study were provided with the opportunity to symbolize the fractions $\frac{5}{8}$ and $\frac{7}{12}$ to represent Andrew and Bella's respective running positions. This aligns with previous research that has highlighted fractions as a crucial concept for developing proportional reasoning skills (Kramer & Funkhouser, 2005; Lamon, 2012).

Proportional reasoning is a fundamental mathematical concept that allows individuals to compare and relate different quantities in various contexts (Carpenter & Moser, 2014; Lambdin et al, 2017; Wang et al., 2019; Ni & Zhou, 2021). Understanding fractions and their symbolic representation is essential for developing proportional reasoning abilities. By symbolizing the fractions to represent the students' running positions, the study provided the students with the opportunity to engage with and develop their proportional reasoning skills.

During this phase of the study, the students made additional observations. They noticed that the location of the fourth water station marked the halfway point and symbolized it as $\frac{4}{8}$. Furthermore, they discovered that the sixth marker coincided with the same location as the fourth water station. This discovery led them to the concept of equivalent fractions (i.e., $\frac{4}{8} = \frac{1}{2} = \frac{6}{12}$). Equivalent fractions are fractions that represent the same value but are written in different forms. The understanding and identification of equivalent fractions are essential skills in mathematics (Fernando et al, 2019), as they enable students to compare, add, or subtract fractions with different denominators. Once students have a strong grasp of equivalent fractions, they can progress to more complex operations such as fraction multiplication.

The incorporation of equivalent fractions into the study builds on prior research highlighting the significance of these concepts in developing students' fractional understanding and mathematical proficiency (Fernando et al., 2019). By exploring equivalent fractions within the context of running positions and markers on the marathon route, the researcher provided students with a tangible and meaningful way to deepen their understanding of these mathematical concepts.

In this study, students used a variety of strategies to determine how many kilometers Andrew and Bella ran, including proportional reasoning, repeated addition, landmark fractions, partial products (fraction decomposition), and the distributive property. They tried to combine several strategies in order for them to find solution in meaningful way and according to their understanding. Proportional reasoning involved comparing the ratios of distances and finding common multiples to calculate the total distance travelled. Landmark fractions, such as $\frac{4}{8}$, was used as fractions of reference to estimate the product of two fractions. Repeated addition involved adding the numerators of the fractions together a certain number of times to find the total distance. Partial products, or fraction decomposition, involved breaking down fractions into smaller components to make it easier to calculate the product. The distributive property was used to decompose fractions and then multiplied them. For instance, S1 attempted to determine the distance between each marker and water station. After determining the length of each marker and each water station, to find how far Bella ran, S1 multiplied $2\frac{1}{6}$, the length of each marker, with 7 (i.e., $2\frac{1}{6} \times 7$). To find the solution, she used distributive property to multiply the fractions. She decomposed $2\frac{1}{6} \times 7$ to $(2 \times 7) + (\frac{1}{6} \times 7)$. Another interesting strategy used by S1 was the way she tried to find $\frac{1}{6} \times 7$. She noticed that $\frac{1}{6} \times 6$ yields 1, and to find $\frac{1}{6} \times 7$, she needed to add $\frac{1}{6}$ with 1 (the result of $\frac{1}{6} \times 6$). Her strategy was in line with the conjecture of students' thinking in the HLT, namely by using distributive property, the students may decompose fractions to ease the calculation.

Another strategy used by a students where they used double number line to show repeated addition. In this study, to find how many kilometers the distance between water stations (for Andrew's case) and between markers (for Bella's case), S2 divided 26 km, the length of marathon route, with 8 (water station) and with 12 (markers).

After that she used repeated addition to find how far Bella and Andrew ran. For instance, for the case of Bella, she found that the distance for each marker was $2\frac{1}{6}$ and then multiplied it with 7, which can be written as $\frac{7}{12} \times 26 = 2\frac{1}{6} \times 7 = 2\frac{1}{6} + 2\frac{1}{6} + 2\frac{1}{6} + 2\frac{1}{6} + 2\frac{1}{6} + 2\frac{1}{6} + 2\frac{1}{6} = 15\frac{1}{6}$. According to Carbonneau et al (2018), using repeated addition can help students develop a deeper understanding of fraction multiplication, as it allows them to visualize the process of multiplying fractions and connect it to real-world situations. This study showed that repeated addition is one of strategies that can be helpful for teaching students how to multiply fraction, especially for multiplication of fraction with natural number.

Generalizing Level

After discovering the solutions for Andrew and Bella's fraction multiplication problems (i.e., for Andrew: $3\frac{1}{4} \times 5 = 16\frac{1}{4}$ and for Bella: $2\frac{1}{6} \times 7 = 15\frac{1}{6}$), the students attempted to find a pattern and generalize the results. Generalization involves recognizing patterns and pursuing more efficient memory use (Linchevski, 1995; Krutetskii, 1976). When the students encountered difficulty with unfriendly fractions, they used landmark fractions to construct partial products using the distributive property and repeated addition. For instance, when S1 encountered the problem of multiplying $\frac{1}{6}$ by 7, and found it difficult to compute, she decomposed it into ($\frac{1}{6} \times 6$) + $\frac{1}{6}$ = 1 + $\frac{1}{6}$ = 1 $\frac{1}{6}$. Similarly, when the students tried to find $\frac{7}{12} \times 26$ (Bella's case), they discovered that multiplying a fraction by a natural number was challenging. To solve this, they determined the distance between markers and water stations first and then used repeated addition strategies to calculate the length of Andrew and Bella's run. The use of repeated addition and landmark fractions to solve fraction multiplication problems has been documented in previous research. For example, Moyer-Packenham et al. (2016) found that repeated addition supported the development of conceptual understanding among elementary school students learning fraction multiplication. Similarly, Siebert et al. (2019) found that using landmark fractions was effective for helping middle school students develop a better understanding of fraction multiplication and division. In this study, it highlights the importance of recognizing patterns and using efficient strategies in solving fraction multiplication problems.

Formalizing Level

In the math congress of the Running for Fun context, the students discovered a pattern that allowed them to multiply a fraction with a natural number by adding 1 to the denominator of the natural number, multiplying the numerators, and multiplying the denominators. This pattern facilitated efficient solutions for complex fraction multiplication problems. Understanding this pattern and being able to apply it in various scenarios is an essential aspect of mathematical formalizing. Formalizing refers to the process of extending generalizations to develop a general procedure, formula, or rule that can be applied across a range of mathematical problems (Hart, 1987).

Research has consistently highlighted the significance of formalizing in promoting mathematical understanding and problem-solving skills. A study conducted by Pearn and Stephens (2016) focused on middle school students and investigated the impact of explicit instruction on formalizing procedures. The findings of their study revealed that when students received specific instruction on formalizing procedures, they were more likely to incorporate generalizations in their problem-solving approaches. This, in turn, led to improvements in their overall problem-solving abilities.

Furthermore, Silver (1997) argued that formalizing plays a vital role in the development of mathematical fluency. Mathematical fluency refers to the ability of students to apply mathematical concepts effectively in practical situations. By formalizing the pattern for multiplying fractions with natural numbers, the students in the Running for Fun math congress were able to develop a rule that could be applied in various fraction multiplication problems. This process contributed to enhancing their understanding of mathematical concepts and improving their problem-solving abilities.

The discovery of the pattern for multiplying fractions with natural numbers during the math congress of the Running for Fun context was a significant step in the students' mathematizing process. By formalizing this procedure, their mathematical abilities were advanced, as they developed a rule that could be utilized across a range of fraction multiplication problems. This underscores the importance of formalizing in promoting mathematical understanding and problem-solving skills.

Abstracting Level

In the Running for Fun context, the final stage of mathematizing is abstraction, which involves identifying and understanding the underlying mathematical structures and relationships. This stage allows students to go beyond specific examples and procedures and recognize the general principles and patterns that govern mathematical concepts. In Worksheet 2 of the Minilesson: Fractions as Operators, students had the opportunity to revisit and extend their understanding of fractions multiplication with natural numbers. Some students, like S2 and S3, relied on the double number line model to solve the problem, while others, like S1, had already abstracted the formula for multiplying fractions with natural numbers that they had formulated collaboratively.

Numerous studies emphasize the importance of abstraction in mathematical thinking and problem-solving. Schifter and Fosnot (1993) argued that engaging in abstraction helps students develop the ability to identify and apply mathematical structures in various contexts, leading to a deeper understanding of mathematical concepts. By recognizing the underlying structures, students can transfer their knowledge to new situations and solve complex problems. Similarly, Lai and Murray (2003) found that students who use abstraction to identify mathematical relationships are better equipped to solve complex problems and transfer their mathematical knowledge to new situations.

In the Running for Fun context, the students had the opportunity to engage in the final stage of mathematizing, abstraction. Through revisiting and extending their understanding of fractions as operators, they developed strategies to solve complex problems and collaboratively formulated formulas. This process allowed them to identify and understand the underlying mathematical structures, ultimately leading to a deeper understanding of mathematical concepts.

Students' Obstacles

In this study, students encountered difficulty when faced with the task of multiplying unfriendly fractions, which are fractions that do not have simple or easily calculable values. To overcome this challenge, students attempted to solve the problem by using landmark fractions to construct partial products through the application of the distributive property. For example, when faced with the problem of multiplying $2\frac{1}{6}$ by 7, students tried to decompose $2\frac{1}{6} \times 7$ into $(2 \times 7) + (\frac{1}{6} \times 7)$.

Another interesting strategy that emerged from one of the students was when she attempted to find the product of $\frac{1}{6} \times 7$. She noticed that multiplying $\frac{1}{6}$ by 6 yields 1, and to find the product of $\frac{1}{6} \times 7$, she needed to add $\frac{1}{6}$ with 1 (the result of $\frac{1}{6} \times 6$). This student's strategy aligned with the conjecture of students' thinking in the HLT, which suggests that students may use distributive strategies to decompose fractions in order to facilitate calculations.

The use of landmark fractions and the application of the distributive property to solve challenging fraction multiplication problems is an interesting finding in this study. This strategy demonstrates how students attempt to simplify complex calculations by breaking them down into more manageable components. By recognizing the relationships between fractions and leveraging known operations, students are able to navigate the complexities of fraction multiplication.

5.1.2. Mathematizing Processes of Training for Next Year's Marathon Context

Modeling Level

The modeling stage in the context of Training for Next Year's Marathon plays a crucial role in facilitating students' comprehension of fractions as part-whole relationships. During this stage, students attempted to interpret 1 circuit/the track as a circle, and in order to locate half of the circuit/track, they needed to divide the minutes required to complete 1 circuit by two.

Visual representation serves as a valuable tool for supporting students in this modeling stage. In this context, students found a visual representation of the context in the form of a circle, which accurately represented the running track. By utilizing this concrete representation, students were able to gain a better understanding of fractions as they relate to a whole. Dividing the circle into equal parts helped students visually perceive how fractions exemplify various quantities or proportions of the whole track. Moreover, this approach allowed students to grasp the concept of half in relation to the track more effectively.

Symbolizing Level

In this study, for instance, as students understood half of 18 (the rate) as $\frac{1}{2} \times 18 = 9$, the modeling process proceeded to symbolize. They also learned that rate refers to minutes per circuit and that to calculate the rate, they must divide the minutes by the number of circuits of track completed.

In the Training for Next Year's Marathon context, students employed a variety of strategies to solve, for instance, the minutes of Rafa, which completed $2\frac{3}{4}$ circuits with rate 30. S3 utilized a circle model to represent the track/circuit and determine the duration of Rafa's ran. This visual representation allowed S3 to easily conceptualize the problem and arrive at the correct answer. S1, on the other hand, preferred to explain her process in writing. She used her idea to find $\frac{3}{4}$ of 30 first and then add the result,

$22\frac{1}{2}$, with 60 (2 times track, 30×2). For S2, she used the formula of multiplication of fraction with natural number which they found from the Running for Fun context.

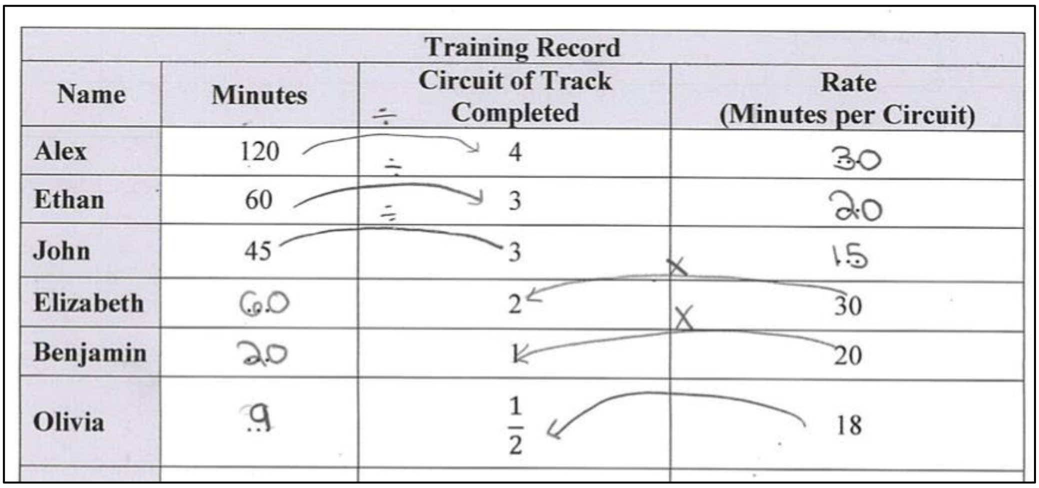
Research has shown that understanding different problem-solving strategies can be beneficial, as it allows students to utilize a variety of approaches to fit their individual learning styles and preferences. In a study by Vlassis & Vitoroulis (2016), they found that utilizing multiple strategies for the same problem increased students' comprehension and ability to solve similar problems in the future. Similarly, Watson and Mason (2006) suggested that exploring various strategies and methods of problem-solving encourages students to engage with the mathematics in a deeper and more meaningful way. Moreover, utilizing a concrete visual model, such as the circle model used by S3 in this context, has been found to be particularly effective for enhancing students' understanding and conceptualization of mathematical ideas. Borasi and Siegel (1994) found that utilizing visual representations encouraged students to think creatively and critically about mathematical concepts, while also promoting problem-solving skills.

Generalizing Level

As the students worked on the next problem, they noticed that the numbers and fractions were carefully chosen, encouraging them to generalize the problem. The students analyzed the rates of Alex, Ethan, and John and discovered that they needed to divide the minutes by the circuit to calculate the rates. In contrast, for Elizabeth, Benjamin, and Olivia, they found that it was necessary to multiply the rate by the circuit. By identifying these relationships between minutes, completed circuits, and rates, the students utilized a ratio table to represent the data. This was in line which the HLT in which "students may use the chart as ratio table in which the arrows illustrate rates relations".

Ratio table was at the heart of this context in which the problem was presented in a table where students needed to understand the relations among minutes, circuit of track completed, and rate (minutes per circuit). Research by Hudson and Miller (2012) found that using ratio tables as a visual tool for solving problems involving fraction multiplication can improve students' conceptual understanding of the procedure. Similarly, Liu and White (2015) found that ratio tables can improve students' procedural fluency and also promote conceptual understanding. In their study, they found that students who used ratio tables were better equipped to apply their knowledge of fraction multiplication to solve complex problems and transfer their skills to new situations. Furthermore, Aliaga et al. (2018) suggested that ratio tables can be particularly beneficial for struggling learners, as they provide a visual representation that supports their understanding of the concepts. In a study by Charalambous and Pitta-Pantazi (2007), they found that using ratio tables helped students to better comprehend the relationships between quantities and become more efficient problem-solvers. Similarly, Wu and Stevens (2019) suggested that the visual representation of ratios and rates through tables can support students in identifying patterns and making predictions. In this study, the Training for Next Year's Marathon context highlighted the effectiveness of utilizing ratio table as a visual representation of rates and relationships between quantities (minutes and circuit of track completed).

During the math congress, students recognized that Alex's running time and distance covered were double that of Elizabeth's, thus their rates were the same: 30 minutes per circuit. This observation exemplifies the application of ratio and proportion in mathematics, which is essential for problem-solving and decision-making in various disciplines. The students utilized multiplication and division to calculate the values of minutes, distance, and rate for each individual. In Figure 5.1, we can see the students' approach to solving the problem by using the concept of proportionality. For example, in the case of Elizabeth, the students may need first to find the minutes, which is 2×30 (= 60). Alex covered the same distance half the time. Then the students will consider $4 \times ? = 120$, and the answer will be 30, as well.



Training Record			
Name	Minutes	Circuit of Track ÷ Completed	Rate (Minutes per Circuit)
Alex	120	÷ 4	30
Ethan	60	÷ 3	20
John	45	3	15
Elizabeth	60	2	30
Benjamin	20	1	20
Olivia	9	$\frac{1}{2}$	18

Figure 5.1. Student's generalization of the relations among minutes, the circuit of track completed, and rate

Formalizing Level

In the formalizing level, students demonstrated an increased understanding of the concept of rate in relation to completing circuit/track. They observed that rate was defined as the number of minutes required to complete one circuit or track. To calculate rate, students realized that they needed to divide the total number of minutes by the number of circuits completed.

This insight aligns with previous research on teaching mathematics, which emphasizes the importance of students grasping the meaning and calculation of rate. For example, studies by Greene et al. (2008) and Archambault and Crippen (2009) highlighted the significance of explicitly teaching the concept of rate by focusing on dividing quantities and quantities per unit. By building on this foundation, students develop a solid understanding of rate as a mathematical concept and can apply it effectively in various contexts, including the Training for Next Year's Marathon scenario.

Abstracting Level

As presented in Figure 4.47 (on page 217), a student in the study discovered "better way" strategy for multiplying fractions by natural numbers. At this stage, she developed her own formula for this operation by using a more abstract approach that incorporated partial products, distributive and associative properties. To illustrate her strategy, she decomposed the fraction $\frac{5}{8}$ into $\frac{1}{8} \times 5$. Using this decomposition, she found:

$$\frac{5}{8} \times 44 = \left(\frac{1}{8} \times 5\right) \times 44 = (\frac{1}{8} \times 44) \times 5 = 5\frac{1}{2} \times 5 = 27\frac{1}{2}$$

The student argued that she could simply divide the natural number directly by the denominator of the fraction, stating, "$\frac{1}{8}$ means 1 part of 8, and since 8 parts is 44, I need to divide 44 by 8 to find 1 part of 8 of 44". This explanation revealed that she employed the concept of fractions as parts of a whole and formulated her own calculation based on this understanding.

The finding that well-chosen examples can elicit generalizations, formalizations, and abstractions from students aligns with the work of prominent mathematicians and education scholars. In his book "Revisiting Mathematics Education: China Lectures", Freudenthal (1991) emphasized the importance of students' active involvement in the process of abstraction through the use of well-chosen examples and the development of mathematical ideas from concrete experiences. Nelissen (1998) also stressed the crucial role of examples in mathematical learning, stating that examples can help students build a shared understanding of mathematical concepts through discussion and reflection. Gravemeijer and Terwel (2000) expanded on the idea of abstraction, proposing that it can be established through the process of horizontal mathematization. This approach to mathematics teaching focuses on building mathematics from real-life situations that contain embedded abstract objects. According to Gravemeijer and Terwel, carefully selected examples could help students understand abstract mathematical concepts by making links between mathematical ideas and real-life situations.

Students' Obstacles

The challenge faced by students in solving the Training for Next Year's Marathon context, particularly concerning Rafa's case with $2\frac{3}{4}$ circuits at a rate of 20, can be examined through the lens of vertical mathematization. The difficulty arose because students once again encountered an unfriendly fraction ($2\frac{3}{4}$). However, some students were able to utilize the algorithm they had learned in the previous problem (the Running for Fun context) to solve the current problem. This highlights the advantage of establishing mathematical connections between different problems and contexts.

Treffers (1987) and Streefland (1991) have discussed the concept of vertical mathematization, which pertains to a higher level of mathematical understanding achieved when students restructure the mathematical system and connect different layers of thinking. According to Treffers (1987), the process of vertical mathematization involves the development and reconstruction of fundamental mathematical concepts, enabling students to progress beyond the mechanical application of procedures and engage in deeper mathematical thinking and reasoning. Streefland (1991) expanded on Treffers' work and proposed that vertical mathematization is accomplished when students establish connections between various mathematical contexts, leading to the construction of a coherent and integrated understanding of mathematical concepts. This process entails linking mathematical operations, procedures, and symbols with real-world situations, ultimately involving mathematical modeling and problem-solving.

The ability of students to apply the algorithm they learned in a previous context to solve the current problem underscores the effectiveness of vertical mathematization in cultivating mathematical thinking and reasoning skills. By building upon prior knowledge and experiences, students can develop a more profound understanding of mathematical concepts and apply their knowledge to tackle novel problems. Notably, this aligns with the notion of mathematical transfer, which refers to the capability of applying previously acquired mathematical knowledge and skills to unfamiliar situations (Lesh & Doerr, 2003; Brousseau et al., 2004). Vertical mathematization offers students opportunities to develop transferable knowledge and skills by establishing connections between different mathematical contexts and problem-solving scenarios.

In conclusion, the difficulty faced by students in solving the Training for Next Year's Marathon context, particularly regarding Rafa's case with $2\frac{3}{4}$ circuits at a rate of 20, can be addressed through vertical mathematization. By connecting various mathematical contexts and problem-solving situations, students can restructure the mathematical system itself, deepen their understanding of mathematical concepts, and apply their knowledge to address novel problems. This process is crucial for fostering flexible and adaptable mathematical thinking and reasoning skills.

5.1.3. Mathematizing Processes of Exploring Playground and Blacktop Areas Context

Modeling Level

In section 4.2.3, the context of Exploring Playgrounds and Blacktop Area allowed students to apply the array model for fraction multiplication to identify the relationship between the blacktopped area and the total area of the lot. This allowed them to gain a deeper understanding of the concept of fraction multiplication. The use of the array model has several advantages, as discussed by Van de Walle et al. (2013).

Firstly, the array model works well in situations where splitting a length or area can be challenging. The model offers visual representations of multiplication that can help students to see how fractions can be combined to form a whole. Secondly, the array model can be used to demonstrate how a result can be significantly less than either of the fractions used. This helps students to understand how multiplication does not always result in a larger number. Finally, the use of the array model can aid in connecting to the standard fraction multiplication algorithm. By understanding the concept visually, students may be better able to apply the steps of the algorithm accurately and efficiently. Van de Walle et al. (2013) emphasized the importance of using multiple models to help students understand mathematical concepts deeply. They recommended that teachers provide various models for fraction multiplication and encourage students to use the model that works best for them.

According to Clements and Sarama (2015), the array model helps students visualize and understand multiplication in a variety of ways. For example, the model can be used to represent the multiplication of two natural numbers as an area of a rectangle, with one dimension representing one of the numbers and the other dimension representing the other number. This model can be extended to represent multiplication of fractions as well, where each fraction represents the length or width of a smaller rectangle within the larger rectangle.

In a study conducted by Morris and Watanabe (2020), the utilization of the array model within the context of Exploring Playgrounds and Blacktop Area proved to be an effective strategy for promoting a deeper understanding of fraction multiplication among students. This model offers several advantages that support students' mathematical comprehension. Firstly, it provides visual representations, allowing students to visualize the multiplication process and better understand the relationship between fractions. Additionally, it demonstrates how the product of two fractions can be smaller than the fractions themselves, helping students grasp the concept of fraction multiplication more fully. Moreover, the array model aids in connecting the visual representation of multiplication to the standard algorithm, facilitating the bridging between different mathematical representations.

In this study, the use of the array model in the context of Exploring Playgrounds and Blacktop Area allowed students to develop a deeper understanding of fraction multiplication. The model's advantages, such as offering visual representations, demonstrating how the result can be less than the fractions used, and aiding in connecting to the standard algorithm, can be advantageous in supporting students' mathematical understanding. Additionally, the model's visual nature aids in making the concept of multiplication more concrete and less abstract.

Symbolizing Level

After engaging in a modeling activity utilizing an array model, the students gained the ability to represent the relationship between the area of the playground and the total area of the lot using symbols. For instance, when presented with the context that the playground occupies $\frac{3}{4}$ of the entire area, the students were able to symbolize this using a rectangle divided into four equal parts, with three parts shaded to denote the playground area. This conceptual understanding of fractions is vital for students to establish a solid groundwork in mathematical concepts.

Research conducted by Heid and Dick (2014) indicated that students often face challenges in comprehending and representing fractions, which can subsequently hinder their progress in mastering more advanced mathematical concepts. To address this issue, they propose using visual models, including the array model, to assist students in constructing a deeper understanding of fractions and their interconnectedness. By employing visual models and symbols, students can bridge the gap between abstract fraction concepts and real-world applications, facilitating the development of a more tangible understanding.

The utilization of symbols as representations of fractions also profoundly contributes to the development of algebraic thinking among students. As Fraivillig and Murphy (2011) argued, symbols such as $\frac{3}{4}$ denoted a relationship between two numbers, similar to how variables in algebra indicate relationships between values. By comprehending and manipulating symbols, students could establish a strong foundation for algebraic thinking and problem-solving.

Generalizing Level

At the stage of generalizing rules, students develop the capacity to make connections and apply rules learned from specific examples to more abstract concepts. This cognitive process of generalization allows students to recognize patterns and similarities across different situations, enabling them to understand and apply mathematical principles more effectively. In this study, it is found that students were able to generalize the meaning of the word “of” as representing multiplication. Consequently, they discovered that expressing expressions such as $\frac{2}{5}$ of $\frac{3}{4}$ (which involves multiplying a fraction by another fraction) could be represented more mathematically as $\frac{2}{5} \times \frac{3}{4}$.

The ability to generalize mathematical concepts is a crucial aspect of students' mathematical development. It equips them with a deeper understanding of mathematical principles and empowers them to apply their knowledge in diverse contexts. When students generalize rules, they are essentially bridging the gap between concrete examples and abstract concepts, enabling them to transfer their knowledge to novel situations. This transfer of learning is an essential skill in problem-solving and higher-order thinking in mathematics. Research by Brown and Jones (2018) emphasized the importance of generalization in enhancing students' ability to reason mathematically and solve complex problems. By recognizing patterns and underlying principles, students can extend their understanding beyond individual examples and develop a more flexible and adaptable mathematical mindset.

Furthermore, the process of generalization not only supports students' mathematical understanding but also fosters their creativity and advanced reasoning skills. Wang and Johnson (2019) highlighted that generalization enables students to engage in abstract thinking and make connections between different mathematical concepts. By recognizing commonalities and features that transcend specific examples, students could uncover more profound and overarching mathematical principles. This ability to think abstractly and make generalizations is a hallmark of higher-level mathematical reasoning and contributes to the development of mathematical creativity.

Formalizing Level

As students progress in their understanding of the multiplication of fractions, the ability to symbolize this operation mathematically represented a level of formalization in their mathematical thinking (Mack, 2010). This level of formalization signified a more advanced stage in their conceptual understanding, where students could represent mathematical ideas using symbols and equations.

Building on their prior knowledge from the activities Running for Fun and Training for Next Year's Marathon, the students were able to generalize the multiplication of fractions using the example $\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{20} = \frac{6}{20}$. This understanding enabled them to represent this mathematical operation symbolically as $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, where $\frac{a}{b}$ and $\frac{c}{d}$ represent fractions (Mack, 2010).

The ability to symbolize the multiplication of fractions mathematically is an essential skill for students to develop. It allows them to manipulate and solve complex mathematical problems involving fractions efficiently. According to Mack (2010), recognizing the pattern and generalizing the multiplication of fractions can enhance students' understanding and proficiency in working with fractional quantities. It provides them with a more systematic approach to calculating and operating with fractions.

The equation $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ demonstrated the application of the commutative and associative properties of multiplication. Students learned that the order of the factors does not affect the product when multiplying fractions. They also discovered that multiplying the numerators together and multiplying the denominators together results in the product of the two fractions.

By understanding the symbolic representation of fraction multiplication, students could apply this knowledge to solve various mathematical problems. They could extend their understanding to real-world scenarios that involve fractional quantities.

Abstracting Level

As students engaged in problem-solving activities, they had the opportunity to uncover patterns and relationships that exist between different mathematical concepts. When working with the multiplication of fractions, students might observe the application of the commutative property. For instance, in this particular study, students discovered that the result of $\frac{2}{5} \times \frac{3}{4}$ is equivalent to $\frac{3}{4} \times \frac{2}{5}$, indicating that the order of multiplication does not affect the outcome. This finding signifies that the expression $\frac{a}{b} \times \frac{c}{d}$ is interchangeable with $\frac{c}{d} \times \frac{a}{b}$. By recognizing such connections, students developed a more profound understanding of mathematical concepts and enhanced their ability to retain and apply new knowledge (Carpenter et al., 2014).

A study conducted by Reid and Knipping (2019) emphasized the significance of fostering a broader, conceptual understanding of mathematical concepts among students. They argued that a deeper understanding of mathematical principles benefits students in their later learning experiences. To promote this understanding, the they proposed encouraging students to explain their reasoning and made connections across diverse problem contexts. By actively engaging in this process, students could develop a more coherent and interconnected understanding of mathematics, enabling them to transfer their knowledge to various situations.

Students Obstacles

As conjectured in the HLT, one student faced challenges in determining the fractional part of the blacktopped area. This difficulty arose because the student employed the same method of partitioning (either vertically or horizontally) when working with fifths and fourths. The student's struggle indicated a lack of comprehension regarding how the partitioning of a rectangle can impact the fractional representation of an area. However, the student managed to overcome this obstacle by utilizing the dimensions of the lot (50 meters by 100 meters). By doing so, the student arrived at the conclusion that both Botany and Gulhane have equivalent blacktopped areas, measuring 1,500 square meters. This study provided evidence supporting the notion that students often encounter difficulties with fractions due to a lack of conceptual understanding of fraction operations and representations (similarly indicated in Babai & Stavy, 2017). In this specific case, the student's struggle underscores the significance of comprehending how fractions can be visually represented through partitioning. This understanding facilitates the connection between fractions and geometric concepts and empowers students to tackle real-world problems more effectively.

Research conducted by Babai and Stavy (2017) reinforced the finding that students' struggles with fractions often stem from a deficiency in their conceptual understanding of fraction operations. They argued that fostering a deeper understanding of fraction concepts and operations is crucial for improving students' mathematical proficiency. Visual representations, such as partitioned rectangles, could aid in developing this understanding by providing students with a tangible connection between fractions and geometry.

The integration of real-world contexts and problem-solving approaches has been found to be an effective strategy for promoting students' understanding, retention of knowledge, and engagement in mathematics (Verschaffel et al., 2014). By connecting mathematical concepts to everyday situations, educators can create a sense of relevance and meaning, which motivates students to learn and apply their mathematical skills in meaningful ways. This approach not only enhances students' conceptual understanding but also equips them with problem-solving strategies that can be transferred to various real-life scenarios.

In conclusion, students often encountered challenges in understanding fractions in relation to geometric concepts. The difficulties arose due to a lack of conceptual understanding of fraction operations, as evidenced in a student's struggle in determining the blacktopped area. However, using partitioned rectangles and real-world contexts could help bridge this gap by visually representing fractions and fostering connections between mathematical concepts and practical applications. By employing problem-solving approaches within real-world contexts, educators can enhance students' understanding, knowledge retention, and engagement in mathematics.

5.1.4. Mathematizing Processes of Comparing the Cost of Blacktopping Context

Modeling Level

In the Comparing the Cost of Blacktopping context, students were introduced to the array model as a visual representation to determine the blacktopped area of the lots. This approach was beneficial in enabling students to understand the problem more concretely and develop effective strategies (Teong et al., 2020). The array model has been widely used in mathematics education to support the understanding of multiplication and division concepts, as well as more complex concepts like fractions, decimals, and percentages (García-Orza et al., 2018). By using the array model, students can identify and compare different quantities, facilitating their understanding of problem-solving strategies.

Furthermore, the students in this activity were introduced to the concept of percentages as they calculated the cost of blacktopping. They learned that percentages represent values relative to 100 and can be expressed using the symbolic notation of %. This understanding allows students to represent different percentages of a quantity and compare them easily (García-Orza et al., 2018). Research has shown that the use of visual representations like the array model and symbolic notation in mathematics significantly support students' learning and understanding in various mathematical concepts (Ng et al., 2016). By building a strong foundation in the array model and percentages, students are better equipped to enhance their mathematical knowledge and skills.

Symbolizing Level

In this study, from modeling the problem, the students were introduced to the concept of percentage, which represents a value relative to 100. To symbolize percentages, the students used fractions, such as $\frac{80}{100}$, to represent 80% of the price. By utilizing this symbolic notation, the students were able to better comprehend the notion of percentages and their relationship to one hundred.

Generalizing Level

During the generalizing level of this study, students went beyond simply understanding the concept of percentages. They delved deeper into the meaning of the word "of" and recognized it as an indication for the operation of multiplication when dealing with percentages. By grasping this crucial connection, students were able to expand their mathematical understanding when solving problems involving percentages.

For instance, when presented with the task of finding 80% of $9, the students skillfully applied their comprehension of the word "of" as multiplication. They transformed the problem into a mathematical expression using the notation $\frac{80}{100} \times \$9$. In this representation, the fraction $\frac{80}{100}$ denoted the equivalent of 80% and the dollar amount $9 signifies the total value. By employing this mathematical technique, students were not only able to find the specific solution to this problem, but they were also empowered to generalize their knowledge and apply it to a myriad of other percentage-related scenarios.

This ability to generalize knowledge is crucial as it enables students to transfer their understanding of percentages to various real-world situations and problem-solving contexts (Kaput et al., 2014). By formalizing their understanding using the notation $\frac{80}{100} \times \$9$, students were equipped with a flexible approach that could be applied consistently across a wider range of problems. This reinforced their comprehension of percentages and provided them with a solid foundation for further mathematical development in the realm of percentages.

Formulazing Level

Building upon their previous knowledge and understanding, the students in the study applied their proficiency in fraction multiplication when solving the problem 80% of $9. They formulated the multiplication process by representing 80% as a fraction ($\frac{80}{100}$) and multiplying it by the dollar amount ($9). This can be further simplified as $\frac{8}{10}$

multiplied by $\frac{9}{10}$, resulting in $\frac{72}{10}$ or $7\frac{2}{10}$, which is equivalent to $7\frac{1}{5}$, or in a more mathematical way: $\frac{80}{100} \times \$9 = \frac{8}{10} \times \frac{9}{10} = \frac{72}{10} = 7\frac{2}{10} = 7\frac{1}{5}$. Through this calculation, the students showcased their ability to manipulate fractions and compute the desired percentage of the given value (Kaput et al., 2014).

Moreover, their keen observations and critical thinking skills led them to an insightful discovery. They noticed that the result derived from $\frac{80}{100} \times \$9$ was equivalent to that of $\frac{90}{100} \times \$8$, as seen below:

$$\$9 \times \frac{80}{100} = \$8 \times \frac{90}{100}$$
$$\frac{9}{1} \times \frac{8}{10} = \frac{8}{1} \times \frac{9}{10}$$
$$\frac{9 \times 8}{10} = \frac{8 \times 9}{10}$$

This observation allowed the students to transcend the specific problem at hand and ascend to a more abstract level of understanding. By recognizing this equality, the students applied the commutative property, which states that the order of multiplication does not affect the outcome, even in the context of percentages. This realization allowed them to generalize their understanding and apply it to various scenarios involving percentages, providing them with a deeper understanding of mathematical principles and properties (Kaput et al., 2014).

Abstracting Level

Furthermore, the students in the study exhibited their deepened understanding of percentages by employing their knowledge of multiplying fractions in this particular context. They recognized that finding a percentage of a given amount entails the multiplication of a fraction and the total value. By utilizing the strategy of multiplying fractions, students were able to apply their algebraic reasoning skills and represent the relationship between the given percentage and the total amount in a more abstract manner (Kaput et al., 2014). This level of abstraction indicated their ability to transfer and apply mathematical concepts across different problem-solving scenarios.

Furthermore, the students astutely observed that the result of multiplying \$9 by $\frac{80}{100}$ was equivalent to the result of multiplying \$8 by $\frac{90}{100}$. This realization prompted them to identify the application of the commutative property of multiplication within this context (i.e., $\frac{a \times b}{c} = \frac{b \times a}{c}$). The commutative property stated that the order of multiplication does not affect the final outcome. By recognizing this connection and understanding how the commutative property operates, students were able to make meaningful connections between different mathematical concepts and properties (Middleton & Sarama, 2020). This ability to relate and integrate various mathematical principles highlights the students' advanced mathematical thinking and their capacity to link different ideas to enhance their overall understanding.

Students' Obstacles

According to the HLT conjecture, it was anticipated that students might encounter difficulties when working with fractions that are considered "unfriendly" in certain mathematical contexts. This concept was exemplified in a specific scenario where a student struggled with determining the cost of blacktopping per square meter. The given cost was expressed as the unfriendly fraction of \$$7\frac{1}{5}$. To calculate the total cost, the student needed to multiply this fraction by the area of blacktopping, which was measured as 1,500 square meters. In order to solve the problem, the student employed the distributive property, recognizing that the fraction $7\frac{1}{5}$ posed a challenge.

By utilizing the distributive property, the student employed an algebraic tool that enables the simplification of complex calculations involving fractions and decimals. This property enables the breaking down of multiplication into simpler operations that are easier to perform. In this particular situation, the student applied the distributive property to expand the multiplication of $7\frac{1}{5}$ by 1,500, resulting in the composition of two separate products: $(7 \times 1{,}500) + \left(\frac{1}{5} \times 1{,}500\right)$. The student then proceeded to simplify each term individually, yielding a total cost of blacktopping amounting to \$10,500 + \$300, which can be expressed as \$10,800.

The use of the distributive property in this example highlights its utility in facilitating mathematical operations and computations involving complex fractions. By decomposing the problem and breaking it down into simpler components, the student was able to navigate the challenges posed by the "unfriendly" fractions and arrived at the desired solution. This demonstrated the usefulness of employing algebraic reasoning and properties to enhance problem-solving abilities in mathematics.

5.2. Conclusion on the Overall Nature of the Study

In this study, the feasibility and effectiveness of learning activities related to fraction multiplication designed within the context of RME were tested and investigated. The study utilized the mathematizing process as a guide to observe and analyze the learning progress in fifth-grade students. The research questions focused on the facilitation of mathematizing processes in students' learning and the obstacles faced by fifth-grade students in learning multiplication of fractions.

The mathematizing process involves five phases: modeling, symbolizing, generalizing, formalizing, and abstracting (Streefland, 1991). Through the learning activities, students were guided through this process to develop a deeper understanding of fraction multiplication. The findings of the study showed that students were able to learn about the multiplication of fractions through mathematizing processes.

The study found that students had a better understanding of multiplication of fractions by focusing on the process of mathematizing. The students initially modeled the problem by using concrete objects to build their understanding of fractions. Students then used symbols to represent their understanding of fraction multiplication. They then generalized their understanding beyond specific problems, formalizing it into a procedure for fraction multiplication. Finally, they abstracted their understanding by applying it to different situations.

However, the study also identified some obstacles that the students faced when learning fraction multiplication. Some students struggled with the abstract nature of the problem and found it difficult to apply the mathematizing process. Additionally, some students had difficulty transferring their understanding of multiplication of natural numbers to fractions.

The modeling phase of mathematizing is a crucial step in the process of learning fraction multiplication. In this phase, students use concrete objects and visual aids to make sense of the problem and develop a deeper understanding of the concept. In the current study, students utilized a number line, ratio table, and array models to comprehend and solve the problems related to fraction multiplication (Streefland, 1991).

The number line model provided a linear representation of fractions, making it easier for students to visualize fraction multiplication. The ratio table model used a table format to organize and compare fractions, allowing the students to see the connections between the fractions. Finally, the array model used a grid structure to represent fractions and multiplication, making it easier to comprehend the concept of multiplying fractions.

The students used these models to solve problems such as Running for Fun, Training for Next Year's Marathon, and Exploring Playgrounds and Blacktop Areas. These problems required the students to use their spatial awareness and logical reasoning to develop an understanding of fraction multiplication concepts.

Previous studies have shown that the use of models in math education can significantly enhance students' understanding of mathematical concepts (Lesh & Doerr, 2003). Moreover, models can help students to identify connections between mathematical concepts and provide a better visualization of the problem (Skemp, 1976). Therefore, integrating models into the teaching of fraction multiplication can be a valuable technique to increase students' learning outcomes. Therefore, in this study, modeling is an essential component of mathematizing processes in learning fraction multiplication, as it provides a bridge between concrete and abstract concepts. The use of models such as a number line, ratio table, and array model had proven to be effective in improving students' comprehension of fraction multiplication.

In RME, models are viewed as essential tools for representing different types of problem situations. However, it is important to note that the mathematical concepts and structures represented in the models are open to interpretation. Therefore, models can take many forms, ranging from physical materials to visual sketches to symbols. As long as they reflect the relevant mathematical concepts in a real-world context, models can be adapted to suit a range of situations (Treffers, 1987, 1991; Gravemeijer, 1994, 1997, 1998, 2001, 2006; Gravemeijer & Terwel, 2000).

To be effective, models used in teaching should possess two critical features. First, they must be rooted in realistic and imaginable settings to help students visualize the problem. Second, they need to be flexible enough to be adaptable to different contexts and situations. Therefore, models need to be able to support not only in specific learning situations but also in a more general context.

In RME, scaffolding strategies are often used to assist students in building a deeper understanding of mathematical concepts. According to Vygotsky's (1978) theory, learning involves the use of higher-level thinking skills, with the support of more knowledgeable others. Scaffolding provides this kind of assistance, enabling students to progress in their learning while giving them the freedom to return to more fundamental levels of instruction if needed.

In the context of RME, models are considered an important tool for facilitating students' active learning. According to RME principles, students should be empowered to construct their own mathematical knowledge, rather than just passively receiving information from teachers (Van den Heuvel-Panhuizen, 2010). Therefore, models used in RME should be designed in a way that enables students to independently reinvent them and adapt them to suit their own problem-solving strategies.

To meet the requirements of RME, models should be flexible and adaptable, so students feel empowered to use them as tools for their own learning. Moreover, models should be designed in a way that integrates with students' informal strategies, making it feel like they constructed the model on their own. This helps learners develop ownership and agency over their learning (Verschaffel et al., 2010).

In this study, the number line model first emerged when the students proposed 'stretching the running route into a line'. The use of well-chosen context, such as that in the Running for Fun task, could help students connect mathematical concepts to the real world, making it more meaningful and easier to understand (Van den Heuvel-Panhuizen & Drijvers, 2014). The context of measuring distance during a run was found to be an effective way of introducing equivalent fractions as being located at the same position on the number line, for instance, $\frac{1}{2} = \frac{4}{8} = \frac{6}{12}$. The findings of this study supported previous research conducted by Petit et al. (2010) and Siegler et al. (2010), which found that using a number line helped students understand fractions and compare them effectively. Holmes and Alcock (2019) found that through the number line model, students were able to compare fractions easily, which is a crucial aspect of understanding fractions.

In summary, the use of a number line in the Running for Fun context was found to be an effective way of facilitating students' understanding of fractions, including equivalent fractions and comparing them. This finding was consistent with prior research that highlighted the usefulness of number lines in teaching fractions.

In the context of Training for Next Year's Marathon, the use of a ratio table emerged as a powerful strategy for students to keep track of comparisons of multiplicative patterns. Streefland (1991) defines a ratio table as a tool that helps learners solve problems involving relations between quantities through proportional reasoning. The ratio table is particularly useful when working with problems involving two or more quantities that are related by multiplication or division. The effectiveness of ratio tables has been recognized in mathematical education research. Van Galen et al. (2008) found that using ratio tables allows students to identify relationships between variables more easily and helps them maintain focus on a single unit of measurement. By using a ratio table, students can recognize proportional relationships between two or more variables. In the Training for Next Year's Marathon context, a ratio table allowed students to compare and record the distances and times ran by different runners. For example, if two runners completed an equal distance but took different times, students could use a ratio table to keep track of the time taken by each runner. Proportional reasoning is essential to accurately completing a ratio table. To summarize, the use of a ratio table is an effective tool for students to keep track of comparisons of multiplicative patterns. In this study, ratio table had been found to help students identify relationships between variables and maintain focus on a unit of measurement. For instance, in the context, it is found that Alex ran twice as many minutes as Elizabeth and completed twice as much distance as Elizabeth, then his rate is the same as Elizabeth's rate: 30 minutes for one circuit. The research showed that proportional reasoning is a fundamental aspect of the use of a ratio table.

In the context of Exploring Playgrounds and Blacktop Areas, students were encouraged to use an array model as a tool to represent and analyze the relationships between different elements of the playground. An array model is a visual representation of the relationships between factors in a multiplication or division problem. In the context of playgrounds, an array model can help students investigate how different elements of the playground relate to each other. Researchers have found that the use of array models can help students develop a deeper understanding of multiplication and division. Martin and Lambert (1990) found that using arrays helped students understand the structure of multiplication problems and the commutative and distributive properties of multiplication.

Similarly, Hong et al. (2013) found that using an array model could promote flexible thinking about problem-solving strategies. In the Exploring Playgrounds and Blacktop Areas context, students used an array model to investigate the relationships between different elements of the playground. For example, they explored how the blacktop area related to the playground and how the playground related to the overall lot. This allowed them to gain a deeper understanding of how different elements of the playground fit together. Overall, the use of an array model in the context of playgrounds can help students develop a deeper understanding of the relationships between different elements of the playground. This study indicated that using array models can promote flexible thinking about problem-solving strategies and deepen students' understanding of fraction multiplication.

In the second level of mathematizing process, students move beyond concrete models to symbolize mathematical concepts and ideas. This is particularly evident in the context of fractions, where students use symbols to represent parts of a whole. In the context of Exploring Playgrounds and Blacktop Areas task, students use the symbols "4" and "3" to represent different parts of the playground and the lot. Linchevski (1995) defines symbolizing as the process of constructing and associating symbols with mathematical concepts. This process involves finding appropriate representations that can be used to communicate mathematical ideas. Gravemeijer (1998) argues that both models and symbols can be used as representations of mathematical ideas. Models can be developed into symbols, and symbols can be combined with other symbols to create more complex representations. In the Exploring Playgrounds and Blacktop Areas context, students used symbols to represent different fractions of the playground. For example, they might use the symbol "$\frac{3}{4}$" to represent the fraction of the playground that covers three-quarters of the lot. This process of symbolizing allows students to communicate their mathematical thinking more precisely and efficiently. Overall, the process of symbolizing was an important milestone in mathematical learning, as it allowed students to move beyond concrete models to represent mathematical concepts and ideas more abstractly. The research suggested that the use of symbols and models can be complementary, with models providing a visual representation of mathematical ideas, while symbols allow for more concise and precise communication of those ideas.

According to Streefland (1991) mathematizing processes, the third level is the generalization of rules. At this level, students become more aware of applying mathematical rules to different situations, looking for patterns and connections between mathematical ideas. This level builds on the previous levels of visualizing and symbolizing, and is an important step in the development of mathematical thinking. In this study, an example of generalizing a mathematical rule is when students recognized that the word "of" in a fraction problem indicates multiplication, for instance $\frac{2}{5}$ of $\frac{3}{4}$ (a fraction of a fraction) can be represented in a more mathematical way as $\frac{2}{5} \times \frac{3}{4}$. They then applied this rule to new situations, such as solving problems that involve finding a fraction of a fraction. This allowed them to represent complex problems in a more efficient and concise manner. Generalizing is an important mathematical activity as it broadens the applicability of rules, mental constructions, and ideas. As noted by Linchevski (1995), this level is primarily concerned with pattern recognition, where students identify commonalities and connections between different mathematical concepts. Krutetskii (1976) suggested that generalizing also entails the search for more efficient memory use, as students develop increasingly sophisticated mental strategies for solving problems. To summarize, the generalization of rules is a key component of mathematical learning, as it allows students to transfer mathematical concepts to new situations and approach problem-solving in a more flexible and efficient manner.

In the fourth level of mathematizing, students began to express mathematical relationships in a more formalized way. This level is known as formalizing, which is an extension of generalizing, according to Gravemeijer (1994). At this level, students identified a general method or rule that can be applied to different mathematical problems. In the Exploring Playgrounds and Blacktop Areas task, the students discovered a pattern when multiplying a fraction by a natural number. They found that by adding 1 as the denominator of the natural number and multiplying its numerator with the numerator of the fraction and the denominator with the denominator of the fraction, they could find the product. In other words, they were able to formalize the process of multiplying fractions by a natural number. Hart (1987) also notes that formalizing involves the development of a universal procedure, rule, or formula that can be applied to a variety of mathematical situations. In this study, the students were able to generate their own formula to multiply fractions by multiplying the numerators and denominators separately, resulting in the formula $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

At the highest level of mathematizing, students became aware of the mathematical object's invariance, enabling them to connect the "upper" and "lower" layers of thinking (Streefland, 1991). This level is characterized by abstraction, as described by Bergeron et al. (1987). In the Exploring Playgrounds and Blacktop Areas task, the students were given fraction problems in minilessons to solve. Through the process of mathematizing, the students were able to apply their skills to solve the problems using the commutative, distributive, and associative properties they had developed. They were using these properties to solve the problems more mathematically, thus demonstrating an abstraction of the concepts they had learned. Streefland (1991) notes that abstraction involves the ability to recognize underlying patterns and principles that transcend specific examples. In this study, the students were able to abstract the properties of mathematics and use them in a sophisticated way to solve problems. Overall, the highest level of mathematizing requires the ability to connect multiple layers of thinking and recognize the underlying patterns and principles that govern mathematical concepts. This level of mathematical thinking is essential for problem-solving and the development of mathematical fluency.

The second research question of the study investigated the obstacles that fifth-grade students face in the mathematizing process of learning multiplication of fractions. From the Running for Fun context, the findings revealed that students had difficulty connecting the fraction representing the position of the marker/water station with the natural number representing the length of the running route. Additionally, students faced challenges in determining the distances between water bottles and markers, which require the conversion of fractions to decimals. Furthermore, the students struggled to work with unfriendly fractions, such as finding $\frac{7}{12}$ of 26 kilometers. This problem was particularly challenging for students since it requires an understanding of subdivisions of fractions and multiplication of fractions. The mathematizing's fourth level, where students learned to formalize mathematical relationships, is relevant to this finding.

In the second context of the study, Training for Next Year's Marathon, fifth-grade students encountered difficulties in multiplying fractions with natural numbers. For instance, they struggled to find how many minutes Olivia ran, given that she ran half of 18 kilometers, representing $\frac{1}{2} \times 18$. Moreover, the students struggled with unfriendly fractions, such as finding $2\frac{3}{4}$ circuits of track completed at a rate of 30 (i.e, $2\frac{3}{4} \times 30$). This challenge aligns with the mathematizing fourth level, where students learn to formalize mathematical relationships. To assist students in overcoming these difficulties, the study recommended using a variety of representations and visual aids, such as circle model and table. Additionally, instructor could use scaffolding techniques to provide support for struggling students and gradually released responsibility as they became more proficient.

In the Exploring Playgrounds and Blacktop Areas context, students struggled with cutting fifths or fourths vertically or horizontally and determining the fractional part of the overlapping part. This difficulty is related to the mathematizing second level of mathematizing process, where students learned to analyze and classify geometric shapes based on their properties. Moreover, students also had difficulties with unfriendly fractions, such as finding $\frac{6}{20}$ of 1,500 m^2 to represent the blacktop area. This challenge aligns with the fourth level of mathematizing process, where students learn to formalize mathematical relationships. Overall, the study emphasized the importance of providing opportunities for students to explore and experiment with shapes and fractions to develop their spatial reasoning and mathematical thinking skills.

In the Comparing the Cost of Blacktopping context, students also encountered challenges with unfriendly fractions when trying to find the solution for fraction multiplication, such as $7\frac{1}{5} \times 1{,}500$. This challenge aligns with the fourth level of mathematizing process, where students learn to formalize mathematical relationships. However, the study found that students were able to overcome this difficulty by decomposing the fraction multiplication and using the distributive property to find the solution. For instance, they decomposed $7\frac{1}{5} \times 1{,}500 = (7 \times 1{,}500) + \left(\frac{1}{5} \times 1{,}500\right)$, which they learned from the first problem context, Running for Fun, and then added the two products: $10{,}500 + 300 = 10{,}800$. This finding suggested that students could apply their previous knowledge and problem-solving skills to new contexts, contributing to their mathematical growth. According to Fosnot and Dolk (2002), for students to effectively mathematize their world, they must be given the freedom to do so in their own ways while learning. They must be encouraged to work on big ideas and fine-tune their problem-solving skills over time. Overall, the study suggested that providing students with opportunities to apply their previous knowledge to new contexts and experiment with different strategies can enhance their problem-solving skills and mathematical understanding.

5.3. Improvement to the HLT

The analysis of the data collected in this study confirmed that the instructional sequence was successfully implemented according to the initial hypotheses set forth in the teaching experiment. An interesting observation was the seamless transition observed within the classroom community as the students moved from the Running for Fun context to the Comparing the Cost of Blacktopping context. This smooth transition between contexts indicated the effectiveness of a meticulously designed instructional sequence in guiding and shaping the students' overall learning experiences.

This particular finding strongly supports the notion that a carefully planned sequence of learning activities can effectively facilitate the smooth transfer of knowledge from one learning context to another. By carefully and intentionally designing the sequence, students are afforded valuable opportunities to build upon their existing knowledge and skill sets. This enables them to establish meaningful connections between prior knowledge and newly introduced information.

Furthermore, the analysis demonstrated that students exhibited the ability to generate multiple models for understanding fractions. These models included number line, ratio table, and array, among others. This finding aligns with the research emphasizing the importance of utilizing visual representations to support students' understanding of fractions (Lamon, 2012). Visual models have been found to provide learners with concrete tools for making sense of abstract mathematical concepts, enabling them to develop a deeper conceptual understanding of fractions.

The analysis also revealed that students were able to grasp key mathematical and big ideas related to fractions. These ideas included the understanding that fractions represent a relation, the significance of the whole in fraction concepts, and the concept of equivalent fractions. The sharing and reinforcement of these ideas through classroom discussions and activities highlights the importance of social interactions and discourse in mathematical learning (Vygotsky, 1978). The opportunity for students to engage in collaborative discussions and participate in meaningful mathematical discourse allows them to construct their understanding through interactions with peers and the teacher.

This study provides valuable insights into the effectiveness of a well-designed instructional sequence in guiding students' learning experiences and facilitating a smooth transition between learning contexts. It also underscores the significance of utilizing visual representations and promoting social interactions in supporting students' understanding of mathematical concepts.

However, the analysis also highlighted some areas for improvement in the instructional sequence. Specifically, it pointed to the need for more attention to be paid to students' prior knowledge before the start of the instruction. It became evident that the instructional sequence could benefit from greater consideration of students' prior knowledge before the commencement of instruction.

By addressing this area for improvement, future instructional sequences can ensure that students have the necessary prerequisites and skills to effectively engage in the learning process. Taking into account students' prior knowledge can facilitate a more targeted and personalized learning experience, enabling students to make meaningful connections between new information and their existing schema. This approach aligns with research emphasizing the significance of activating prior knowledge as an effective instructional strategy (Bransford, Brown, & Cocking, 2000).

It is important to consider these findings when designing future instructional sequences to ensure that students have the necessary prerequisites and skills to effectively engage in the learning process. Additionally, the study generated valuable insights into students' strategies for problem-solving with fractions. By identifying these strategies, they could be integrated into the HLT to improve and enrich the teaching of fractions. The findings revealed several new strategies that could be added to the current HLT.

Prior knowledge plays a critical role in students' ability to learn new concepts and ideas. In the study, the researcher added additional prior knowledge to the HLT for the first context (Running for Fun) to better prepare students for the problem at hand. The prior knowledge included fraction concepts such as part of a whole and the need to divide the whole into equal parts. Additionally, drawing representations of fractions, simplification of fractions, fraction addition with same denominators, and multiplication as repeated addition were introduced to the students and needed to be added in students' prior knowledge.

Previous research has highlighted the importance of providing students with the necessary prior knowledge to effectively engage in the learning process. For instance, Learmonth et al. (2020) underline the importance of prior knowledge in learning mathematics and suggest that it is essential to assess and provide students with the necessary prior knowledge before introducing new concepts. Similarly, Wood & McNeal (2019) discuss the significance of activating prior knowledge in learning fractions and argue that students are likely to learn better when new concepts build on their existing knowledge.

In addition to the added prior knowledge, the study also identified several new strategies that were developed by the students when they solved the Running for Fun task. For instance, the students used the equivalence of fractions to find halfway points. They also used the distributive property to multiply fractions and observed that to find $\frac{1}{6} \times 7$, they needed to add $\frac{1}{6}$ to 1, which is the result of $\frac{1}{6} \times 6$. These strategies were incorporated into the HLT to enrich the learning experience for future students.

Research has shown that the integration of various strategies into the instructional design can improve student learning outcomes (Hiebert & Lefevre, 1986; Orellana, Garcia-Sanchez, & Flores-Mejia, 2018). For example, Hiebert and Lefevre (1986) found that students who were introduced to multiple strategies for solving division problems performed better than those who were only taught the standard algorithm. Similarly, Orellana et al. (2018) observed that students who were provided with multiple strategies for solving fraction problems performed better than students who were not.

In the second context of the study, the Training for Next Year's Marathon context, the researchers incorporated similar prior knowledge as in the Running for Fun context. Additionally, the reseacher added a circle model to represent fractions as students found it useful in visualizing the relationships between minutes, circuits completed, and rate. The use of visual aids such as the circle model is consistent with previous research on the topic. Various studies have found that visual models, like number lines and fraction bars, can be useful in improving students' understanding of fractions and mathematical concepts (Hiebert et al., 1997; Scherer et al., 2016). The improved HLT also included new student strategies such as using division to find the rate of the runners, multiplication to find the minutes of certain runners, employing formulas discussed earlier, using a circle model to represent the track, and recognizing the fastest runner as the one who takes the least amount of time to complete one track. The incorporation of student-generated strategies is in line with research suggesting that such strategies can promote flexible thinking and enable students to demonstrate their understanding in ways that make sense to them (NMAP, 2008; Stein et al., 1996).

In the third context of the study, which focused on Exploring Playgrounds and Blacktop Areas, the researcher introduced two additional areas of prior knowledge: how to shade fractional parts and the calculation of area for rectangles. The use of visual aids, such as shading fractional parts, is in line with research suggesting that visual representations can provide a valuable tool in teaching mathematical concepts (Leikin, 2012; Hiebert et al., 1997). In addition, the improved HLT incorporated new student strategies, including first finding the total area of the lot or garden before calculating the area of various sections. For example, when working on the Botany garden problem, students first calculated the area of $\frac{3}{4}$ of the lot by multiplying $\frac{3}{4}$ by the total area of 5,000 square meters. They then used division to find the area of the blacktop section, which was determined to be 3,750 square meters.

In the last context of the study, Comparing the Cost of Blacktopping, the researcher introduced two additional areas of prior knowledge, namely the concept of percentage and simplification in percentage calculations. The improved HLT approach incorporated additional students' strategies. One such strategy involved finding the price of blacktopping per square meter after taking into account the discount for each garden. Students could then calculate that the cost of blacktopping per square meter was the same for both Botany and Gulhane gardens, and then multiply by the area of blacktopping to determine the total cost. Another strategy that students foundwas the commutative property to demonstrate the equivalence of $\$9 \times \frac{80}{100} = \$8 \times \frac{90}{100}$. Additionally, students applied the concept of interchanging numerators or denominators to derive the product of two fractions when working on the Minilesson titled Interchanging Numerators.

Overall, the incorporation of additional prior knowledge and new strategies into the improved HLT has the potential to enhance students' learning and improve their ability to solve fraction problems, especially fraction multiplication, effectively. Incorporating students' strategies into mathematics instruction can be a valuable approach for enhancing students' learning and understanding of mathematical concepts. Future research can focus on exploring the different strategies that students use to solve problems and identifying the most effective strategies for improving student learning. This can be achieved by conducting case studies and surveys to understand students' thought processes and the strategies they use to solve problems. Furthermore, incorporating students' strategies into mathematics instruction can have positive effects beyond just improving student learning. For instance, it can make students feel more engaged and invested in their learning, which can lead to increased motivation and higher academic achievement (van der Werf, Kuyper, & van den Bergh, 2013). It also promotes a more inclusive learning environment that recognizes and values students' diverse perspectives and experiences. Additionally, the focus on using real-world contexts and problem-solving strategies is supported by research indicating that these types of activities can enhance student engagement and understanding in mathematics (Lubienski & Turner, 2013; Zodik & Friedlander, 2004). The improved HLT from phase 2 is presented in Table 5.1.

Table 5.1. Improved HLT for learning multiplication of fractions after teaching experiment (phase 2)

Activities • **Running for Fun** • **Math Congress** • **Minilesson 1: Fractions as Operators**	
Learning Goals and Processes	• Students will develop several big ideas related to multiplication with fractions. • Students will share their ideas in solving problem especially in decomposing numbers and using partial products. • Students will revisit their strategies discussed in the math congress and use the idea of decomposing numbers and partial products to solve the string of Minilesson 1.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • To maintain equivalence, the ratio of the related numbers must be kept constant • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Prior Knowledge Required	• Fraction concept as 'part of a whole' and it should be divided into equal parts • Drawing representation of fraction. • Equivalent fractions • Simplification of fraction • How to change improper fractions to mixed fractions, and vice versa • Fractions addition with same denominators • Multiplication as repeated addition
Model for Fraction	(Double) number line

Students' Possible Strategies
• Students notice that the fourth water station means half of 8 water stations. • Students notice that $\frac{7}{12}$ and $\frac{5}{8}$ is more than a half. • Students notice that Andrew ran better than last year because he passed the halfway point, so did Bella. • Students use equivalence of fractions to find $\frac{6}{12}$ which represented halfway of Bella last year and $\frac{4}{8}$ for halfway of Andrew. • To find how many kilometers Andrew ran, students use the idea of proportional reasoning and decomposing number. • First, they try to find a half of 26, which is 13, then find $\frac{1}{4}$ of 13 which is $3\frac{1}{4}$. As they decompose $\frac{5}{8}$ into $\frac{4}{8}+\frac{1}{8}$, means that he needed to add up 13 kilometers with $3\frac{1}{4}$ to find $\frac{5}{8}$ of 26 kilometers. • Another strategy which uses the idea proportional reasoning and decomposing number is that students divide 13 first by 2 which made $6\frac{1}{2}$, then divided $6\frac{1}{2}$ again by 2, which is $3\frac{1}{4}$, to find the distance from one water bottle to the next water bottle. Then, they added 13 kilometers with $3\frac{1}{4}$ to find the length of Andrew ran this year. • Student first try to find how many kilometers for each marker and each water station. • After finding the length of each marker and each water station, to find how far Bella ran, students multiplied $2\frac{1}{6}$, the length of each marker, with 7 ($2\frac{1}{6} \times 7$). • Students use distributive property to multiply the fractions. She decomposed $2\frac{1}{6} \times 7$ to $(2 \times 7) + (\frac{1}{6} \times 7)$. • When students try to find $\frac{1}{6} \times 7$, they noticed that $\frac{1}{6} \times 6$ yields 1, and to find $\frac{1}{6} \times 7$, they needed to add $\frac{1}{6}$ with 1 (the result of $\frac{1}{6} \times 6$). • Students use double number line as guided. • Students divide 26 km, the length of marathon route, with 8 (water station) and with 12 (markers). After that they used repeated addition to find how far Bella and Andrew ran. • Students use the length of half-way route (13 km) instead of the whole length of the route (26 km) to find the distance between water stations and between markers.

- To find how far Andrew ran, students notice that they need to divide 13, the length of half-way of marathon route, with 4, obtained from the 4th water station (half of the marathon route).
- Student come up with the following multiplication of fraction as they realize that Andrew stopped running in the 5th water station.

  $\frac{13}{4} \times 5 = \frac{65}{4} = 16\frac{1}{4}$ km
- Students use same strategy as above to find how far Bella ran. He first divided 13 with 6 (obtained from the 6th marker, half of the marathon route) and then multiplied it with 7 as Bella stopped at the 7th marker.

  $\frac{13}{6} \times 7 = \frac{91}{6} = 15\frac{1}{6}$ km
- Students use proportional reasoning and decomposing numbers to solve multiplication of fractions with natural number in Minilesson 1: Fractions as Operators.

Students' Obstacles

- Students first struggle to connect the fraction (representing the position of market/water station) with natural number (length of running route).
- Students struggle to determine how many kilometers the distance between the water bottles, as well as how many kilometers the distance between markers.
- Students struggle to work with unfriendly fraction, namely to find $\frac{7}{12}$ of 26 kilometers.
- Students struggle to multiply unfriendly fractions.

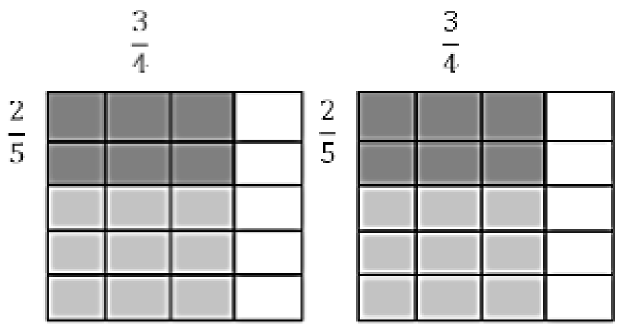
Activities

- **Training for Next Year's Marathon**
- **Math Congress**
- **Minilesson 2: Fractions as Operators**

Learning Goals and Processes	• Students will use landmark fractions and partial products when multiplying a fraction by a natural number. • Students will explore multiplication and division (a natural number by a fraction) and the relationship between the operations.

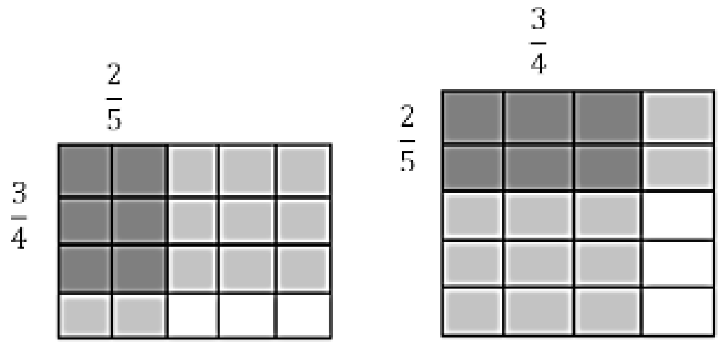
	• Students will share their ideas in solving problem in Activity 3 especially about the patterns on the chart and ideas related to multiplication and division with fractions.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • To maintain equivalence, the ratio of the related numbers must be kept constant • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Prior Knowledge Required	• Fraction concept as 'part of a whole' and it should be divided into equal parts. • Drawing representation of fraction. • Equivalent fractions • Simplification of fraction • How to change improper fractions to mixed fractions, and vice versa • Fractions addition with same denominators • Multiplication as repeated addition
Model for Fraction	Ratio table, (double) number line, and circle model
Students' Possible Strategies	
• Students may find that in order to divide (by fractions), they are multiplying (by the multiplicative inverse of fraction) to find the rate. • Some students may notice the numbers relationships and use proportional reasoning to complete the chart. Proportional reasoning is the heart of this problem in which equivalence will be maintained when the ratio kept constant. For instance, the students might see since Alex ran twice as many minutes as Elizabeth and completed twice as much distance as Elizabeth, then his rate is the same as Elizabeth's rate: 30 minutes for one circuit. • Partial quotients may also be used, for instance for the case of Isabella, she did $\frac{3}{4}$ of the circuit in 15 minutes. Then, $\frac{3}{4}$ will be divided into 3 pieces and each piece was 5 minutes. So, the whole is 20 which we add 15 and 5, because that is $\frac{3}{4} + \frac{1}{4}$.	

• Some may use the relationship of multiplication and division, for instance in the case of Elizabeth, the students may find the minutes first namely 2×30 is 60. Alex ran twice as far. Then they will think about $4 \times ? = 120$, and the result will be 30, too. • Some students may use the double number line model as they try to connect with the previous context, Running for Fun. • Others may use the chart as ratio table in which the arrows will show the relations of the rates. • To find the rate of Alex, Ethan and John, students divide the minutes with the circuit of track completed. • To find the minutes of Elizabeth and Benjamin, students will multiply rate with circuit of track completed. • Students employ the formula discussed during the math congress from previous activity Running for Fun to find "Half of 18", namely adding 1 to the denominator of 18 (i.e., $\frac{1}{2} \times \frac{18}{1} = \frac{18}{2}$). • Students use circle model to represent the track/circuit. • Students notice that the fastest runner is the one with little time to complete 1 track.
Students' Obstacles
• Students face a difficulty to find how many minutes Olivia ran as they are facing with multiplying fraction with natural number, namely $\frac{1}{2} \times 18$. • Student may face difficulty to see in the case of Olivia and Emma as their rates are differ. It is expected that the students will draw the track and mark out the fractions on the circuit if they struggle with Olivia and Emma's case. • Students may struggle to work with unfriendly fraction, i.e., $2\frac{3}{4}$ (circuit of track completed) of 30 (rate) ($2\frac{3}{4} \times 30$).

<table>
<tr><td colspan="2">Activities
• Exploring Playgrounds and Blacktop Areas
• Math Congress
• Minilesson 3: Multiplication of Fractions</td></tr>
<tr><td>Learning Goals and Processes</td><td>• Students will solve the problem related to multiplication of fractions.
• Students will share their ideas in solving problem in Activity 5.</td></tr>
<tr><td>Mathematical/Big Ideas</td><td>• Fractions represent a relation
• The whole matters
• The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers</td></tr>
<tr><td>Prior Knowledge Required</td><td>• Representing fraction in drawing as ‘part of a whole’
• Fractional parts (how to shade the part(s))
• The area of rectangle</td></tr>
<tr><td>Model for Fraction</td><td>Array / area model</td></tr>
<tr><td colspan="2">Students’ Possible Strategies</td></tr>
<tr><td colspan="2">• Students will cut both lots fourths vertically and fifths horizontally and then shade or color the parts in which indicate the blacktopped area, $\frac{3}{4}$ of $\frac{2}{5}$ or $\frac{2}{5}$ of $\frac{3}{4}$. With this strategy, the ratio of the array of the blacktopped area (3 × 2) to the array of the lot (4 × 5). So, the blacktopped area of two lots will congruent.


• Students will cut one lot into fourth horizontally then shade $\frac{3}{4}$ indicating the playground, then marking fifths of that area vertically and shade $\frac{2}{5}$ of it the show</td></tr>
</table>

the blacktop area. Similar way for the other lot, students will first cut the lot into fifths horizontally then shade $\frac{2}{5}$ indicating the playground, then marking fourths of that area vertically and shade $\frac{3}{4}$ of it the show the blacktop area.



- Students perhaps cut the fifths or fourths vertically (or horizontally).
- Students may use the dimensions of the lots, which is 50 meters × 100 meters. This strategy may give result 37.5 meters × 40 meters, and 75 meters × 20 meters. Using this strategy, the students will find that the areas are equivalent but not congruent.
- Students first try to find the area of the lot/garden, namely $50 \times 100 = 5{,}000\ m^2$. For Botany, they first try to find $\frac{3}{4} \times 5{,}000 = \frac{15{,}000}{4} = 3{,}750\ m^2$ and then calculate $\frac{2}{5} \times 3{,}750 = \frac{7{,}500}{5} = 1{,}500\ m^2$. Then, they find that the blacktop area in Botany garden is $3{,}750\ m^2$.
- The same strategy is used by students to find the blacktop area of Gulhane garden. They first calculate the playground area from $\frac{2}{5} \times 5{,}000 = \frac{10{,}000}{5} = 2{,}000\ m^2$ and then calculate $\frac{3}{4} \times 2{,}000 = \frac{6{,}000}{4} = 1{,}500\ m^2$. They find that the blacktopped areas in both gardens are the same as they have the same area, namely $1{,}500\ m^2$.

Students' Obstacles

- When students cut the fifths or fourths either vertically (or horizontally), they will find overlapping part but they might struggle to determine the fractional part.
- Students may be confronted with the problem of finding a means to compare areas that are similar but not congruent.
- Students may struggle to work with unfriendly fraction, i.e., $\frac{6}{20}$ (fraction representing the blacktop area) of 1,500 m^2 (total area of one lot).

Activities • **Comparing the Cost of Blacktopping** • **Math Congress** • **Minilesson 4: Interchanging Numerators**	
Learning Goals and Processes	• Students will use array model to solve the multiplication of fractions problem. • Students will extend their investigation with the commutative property of multiplication of fractions to percentages and decimals. • Students will use interchanging numerators (or denominators) to derive the product of two fractions. • Students will share their ideas in solving problem in Activity 8 especially related to the commutative property and associative property of multiplication with fractions, percentages and decimals.
Mathematical/Big Ideas	• Fractions represent a relation • The whole matters • The properties (distributive, associative, and commutative) that hold for natural numbers, also apply for rational numbers
Prior Knowledge Required	• Concept of percentage (per cent or per hundred) • Simplification in percentage • How to convert percentage into fraction
Model for Fraction	Array/Area Model
Students' Possible Strategies	
• Before comparing the cost, the students will calculate the area of blacktopping for each lot. • The calculation depends on their drawing from previous activity, whether they will calculate $\frac{3}{4}$ of 50 and $\frac{2}{5}$ of 100; or $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100. In calculating $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100, students will probably not have difficulty than calculating $\frac{3}{4}$ of 50. In calculating $\frac{3}{4}$ of 50, students might start with what they know, for instance by	

first find $\frac{1}{2}$ of 50 (=25) and $\frac{1}{2}$ of 25 (=12.5) which is same as $\frac{1}{4}$ of 50, and then they can calculate $\frac{3}{4}$ of 50 as 37.5 meters.

- Students might find the commutative properties that underlies the relationship of both blacktopping in two lots:
  37.5 meters × 40 yards = 75 meters × 1,500 yards = 1,500 square meters.
- After students find the blacktopping areas of two lots, students may multiply the area with \$9 per square meters and offers 80% of the price, students might multiply \$9 by 1,500 to determine the full price of blacktopping which is \$13,500; then to find 20% of \$13,500, students will use landmark of fraction in which 20% equal to $\frac{1}{5}$ and then calculate $\frac{1}{5}$ of \$13,500 (the discount, by dividing by 5, \$2,700); and subtract that discount to get the final price of \$10,800. Similarly, to find the price of blacktopping in other lot uses \$9 per square meters and $\frac{1}{10}$ to calculate the discount.
- Some students will directly include the discount in the calculation and use decimal or fraction forms to show the percentages (i.e., 80% = 0.8 = $\frac{8}{10}$; 90% = 0.9 = $\frac{9}{10}$). Then, calculate $0.8 \times \$9\ \times\ 1{,}500\ square\ meters$ for the cost of first blacktopping area and calculate $0.9 \times \$8\ \times\ 1{,}500\ square\ meters$ for the cost of second blacktopping area. To find the calculation, the students might decompose the percentages or fractions through associative property:

$$\left(8 \times \frac{1}{10}\right) \times 9 \times 1{,}500$$

$$= 8 \times (\frac{1}{10} \times 9) \times 1{,}500$$

- Other students may think that there is no need to include the area as they found it is equivalent. They may just use $0.8 \times \$9$ and $0.9 \times \$8$. In this case, they might be challenged about why 80% of 9 = 90% of 8. It is expected that the students will find the associative property underlies the equivalence:

$$8 \times \left(10 \times \frac{1}{100}\right) \times 9$$

$$= 9 \times \left(10 \times \frac{1}{100}\right) \times 8$$

- Students try to find the price for blacktopping per square meters after discount (i.e., 80% of \$9 for Botany and 90% of \$8 for Gulhane), and that they will find that the cost of blacktopping per square meter in Botany and Gulhane gardens is

the same, $\$7\frac{1}{5}$. To find the total cost, they just need to multiply $\$7\frac{1}{5}$ by the area of blacktopping, $1{,}500\ m^2$. • Students may find that commutative property underlies the equivalence: $\$9 \times \frac{80}{100} = \$8 \times \frac{90}{100}$ $\frac{9}{1} \times \frac{8}{10} = \frac{8}{1} \times \frac{9}{10}$ $\frac{9 \times 8}{10} = \frac{8 \times 9}{10}$ • As students working on the Minilesson "Interchanging Numerators", they may come up with the idea about interchanging numerators (or denominators) to derive the product of two fractions. • The students first use multiplication of fractions formula to solve the problem and then simplified the result (i.e., $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15} = \frac{2}{5}$).
Students' Obstacles
• Student may directly multiply 80% by \$9 as they connect the meaning of word "of" from previous activity as multiplication. • Students have a difficulty in converting percentage into fraction. • Students may struggle to work with unfriendly fraction, i.e., $7\frac{1}{5}$ (80% of \$9, $\frac{80}{100} \times \$9$) of 1,500 m^2 (the area of the lot).

5.4. Visualization of Mathematizing Processes in Learning Multiplication of Fractions Using the Iceberg Metaphor

In addition to the improved HLT developed for learning multiplication of fractions, this study also employed the Iceberg Metaphor to visualize the mathematizing processes involved. The Iceberg Metaphor serves as a powerful tool within the field of education, specifically in the context of mathematics, to illustrate the hidden complexities and underlying conceptual understanding of mathematical concepts.

By employing the Iceberg Metaphor, the researcher aimed to depict the multifaceted nature of mathematical learning and the importance of delving beyond surface-level understanding. The metaphor highlights that mathematical concepts contain both visible and invisible components, with only a fraction of the understanding being visible above the waterline.

This visual representation of the mathematizing processes within the study enables educators and researchers to recognize the significance of uncovering the hidden dimensions of mathematical understanding. By focusing on the invisible part of the iceberg, educators can emphasize the development of deep conceptual understanding rather than merely rote memorization or calculation-based knowledge.

The use of the Iceberg Metaphor also highlights the need for educational interventions, such as the improved HLT system developed in this study, to prioritize the exploration of mathematical connections and reasoning. By incorporating real-life contexts and promoting meaningful mathematical tasks, students can engage actively in the mathematizing process and develop a more profound and flexible understanding of arithmetic operations, such as fraction multiplication.

In the context of RME, the iceberg metaphor illustrates a key concept regarding mathematical understanding and problem-solving. The metaphor suggests that mathematical concepts consist of visible and invisible components, much like an iceberg where only a small portion is visible above the waterline while the majority remains hidden beneath the surface.

According to RME theorists, such as Freudenthal (1991), the visible part of the iceberg represents the procedural or calculation-based knowledge that students often acquire in traditional mathematics instruction. This includes the memorization of facts and procedures without a deep understanding of their underlying concepts.

However, RME emphasizes the importance of uncovering the deeper, invisible part of the iceberg, which represents the conceptual understanding of mathematics. This involves making mathematical connections, exploring relationships, and understanding the reasoning behind procedures. By focusing on the hidden iceberg, students are encouraged to develop a more profound and flexible understanding of mathematical concepts (Drijvers et al., 2010).

The iceberg metaphor in RME underscores the need for educators to delve beyond surface-level learning and guide students towards a deeper comprehension of mathematical principles. By engaging students in meaningful mathematical tasks, incorporating real-life contexts, and encouraging them to reason and explain their mathematical thinking, RME aims to reveal the latent conceptual understanding that lies beneath the surface.

In this study, the mathematizing processes of four different contexts were visualized in Figures 5.2 to 5.5. These contexts, namely Running for Fun, Training for next Year's Marathon, Exploring Playground and Blacktop Areas, and Comparing the Cost of Blacktopping, were carefully designed to facilitate the transition of students learning from a *model-of* context situation to a *model-for* more formal mathematics, and from horizontal mathematization to vertical mathematization. Throughout this transition, various representations such as number line, ratio table, and array/area models were employed to bridge the gap between concrete/informal level understanding and abstract/formal level understanding.

Figures 5.2 to 5.5 serve as visual aids that depict the progression of the mathematizing processes within these four contexts. They illustrate how the use of different mathematical representations enables students to make connections between the real-life contexts and mathematical concepts. The number line representation, for instance, provides a concrete visual tool for understanding mathematical relationships (i.e., fractions represent a relation, relation among fractions, equivalent fractions) and operations (i.e., multiplication of fraction with natural number) within the given contexts (Richland, Holyoak, & Stigler, 2004).

Furthermore, the inclusion of ratio tables and array/area models in the visualization of the mathematizing processes highlights the importance of multiple representations in supporting students' transition towards more formal mathematics. These representations allow students to explore patterns, make connections, and develop a deeper understanding of mathematical concepts (Sfard, 1991).

Through the use of these visualizations, educators and researchers can gain insights into the effectiveness of the transition process and the impact of different representations on students' mathematical understanding. By providing a clear visual pathway from concrete/informal levels to abstract/formal levels, this study contributes to the field of mathematics education by offering a practical framework for supporting students' mathematical development.

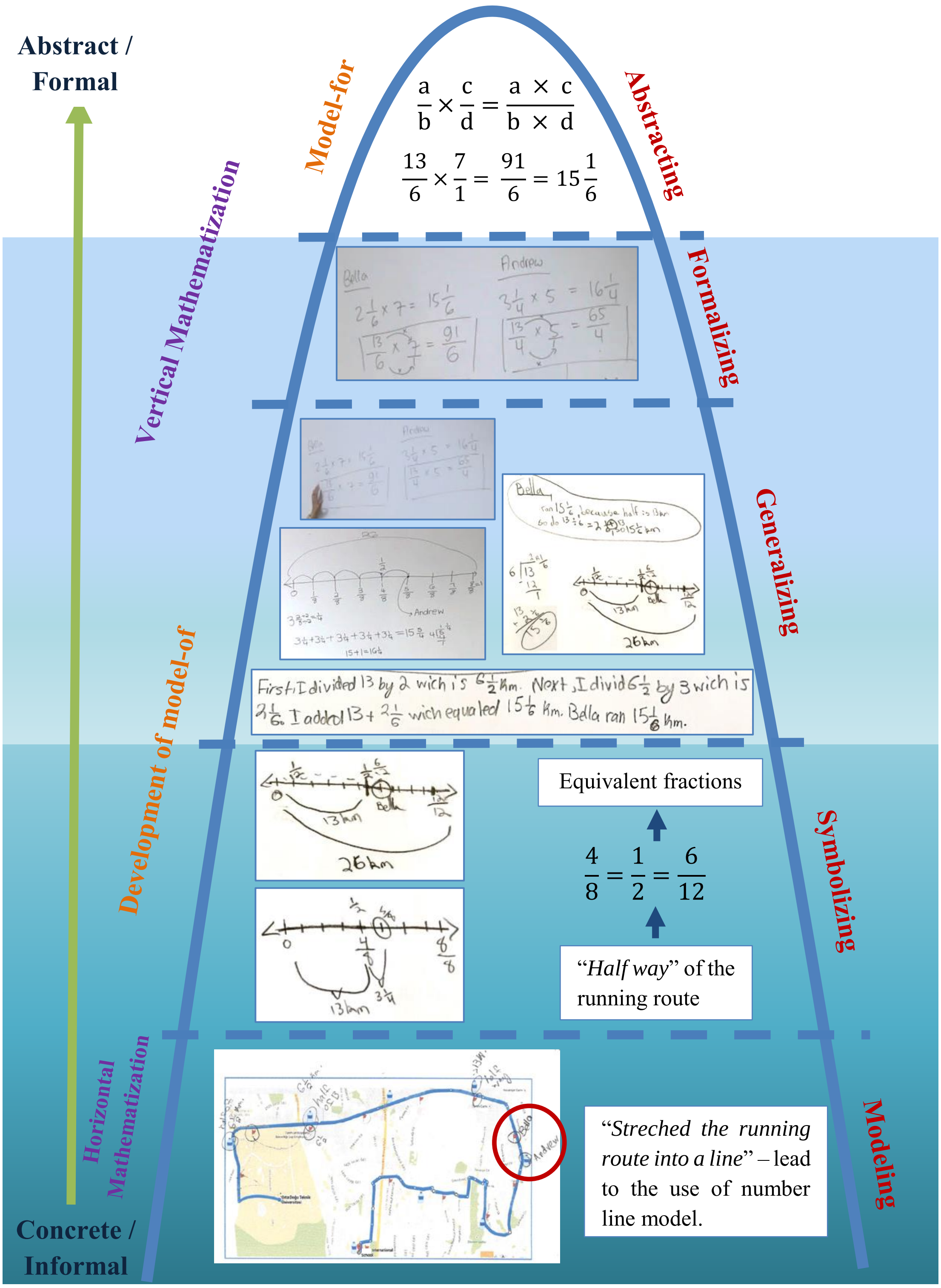


Figure 5.2. Mathematizing Process of Learning Multiplication of Fraction with Natural Number in Iceberg Metaphor – Running for Fun Context

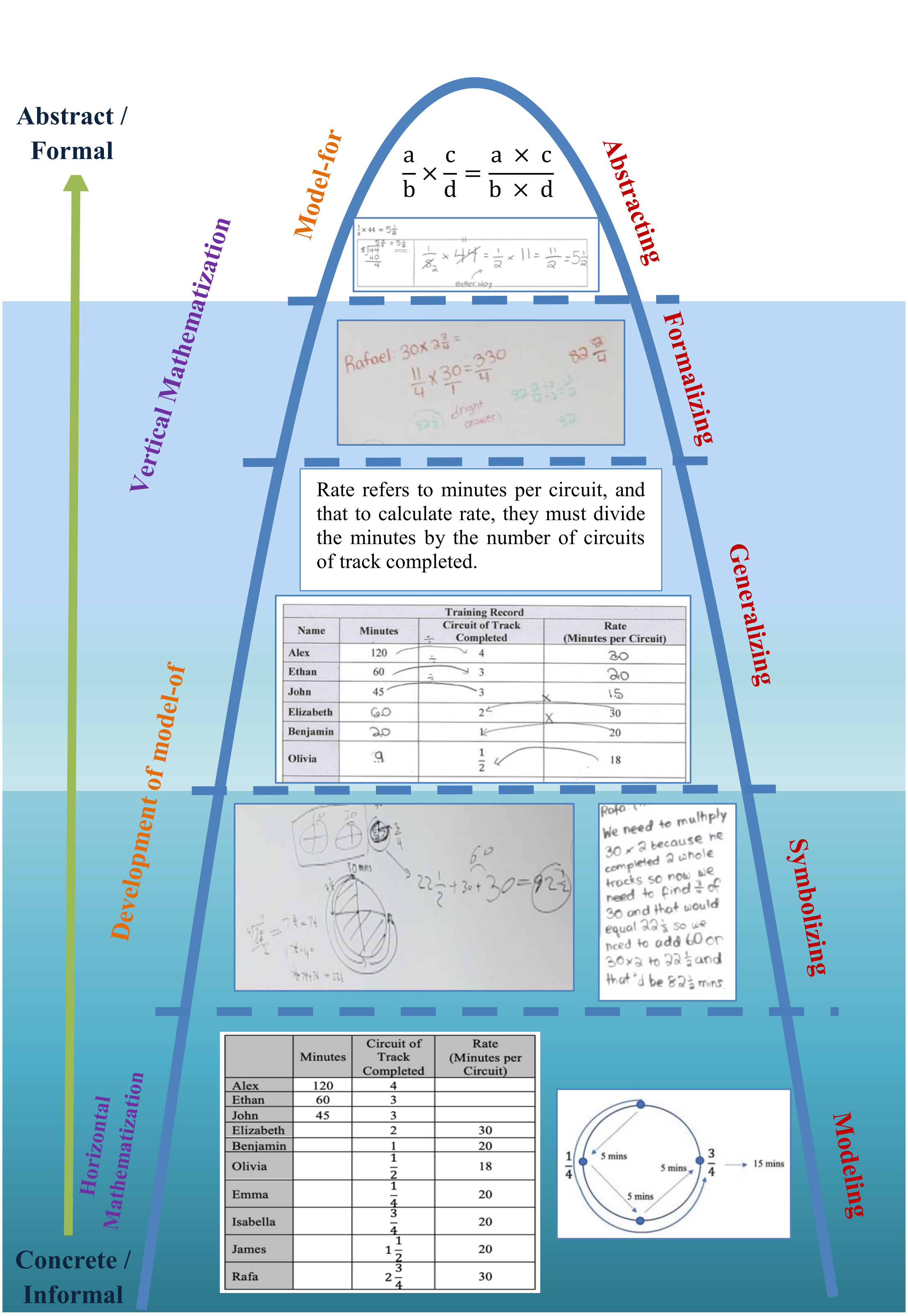


Figure 5.3. Mathematizing Process of Learning Multiplication of Fraction with Natural Number in Iceberg Metaphor – Training for Next Year's Marathon Context

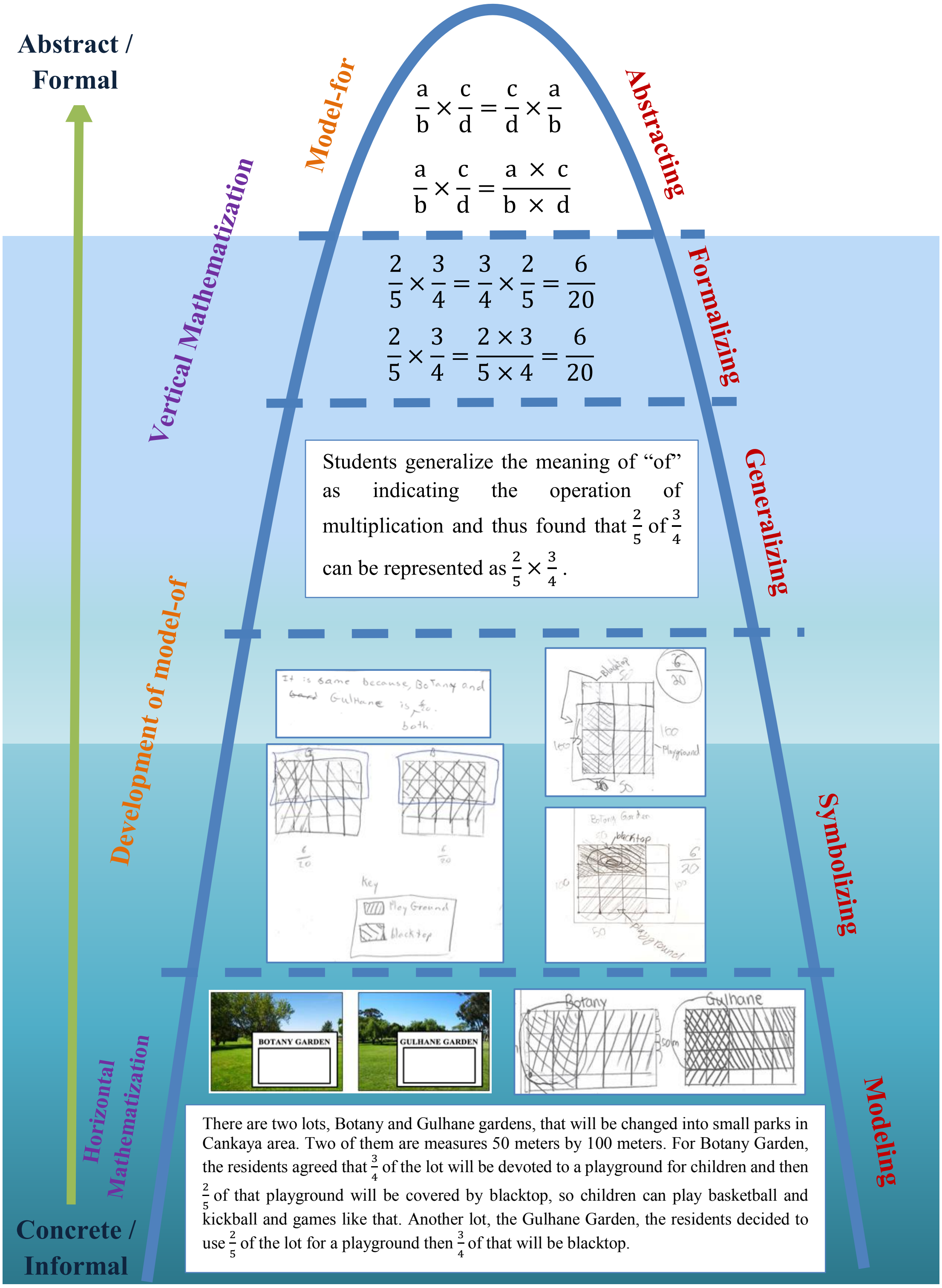


Figure 5.4. Mathematizing Process of Learning Multiplication of Fractions in Iceberg Metaphor – Exploring Playground and Blacktop Areas Context

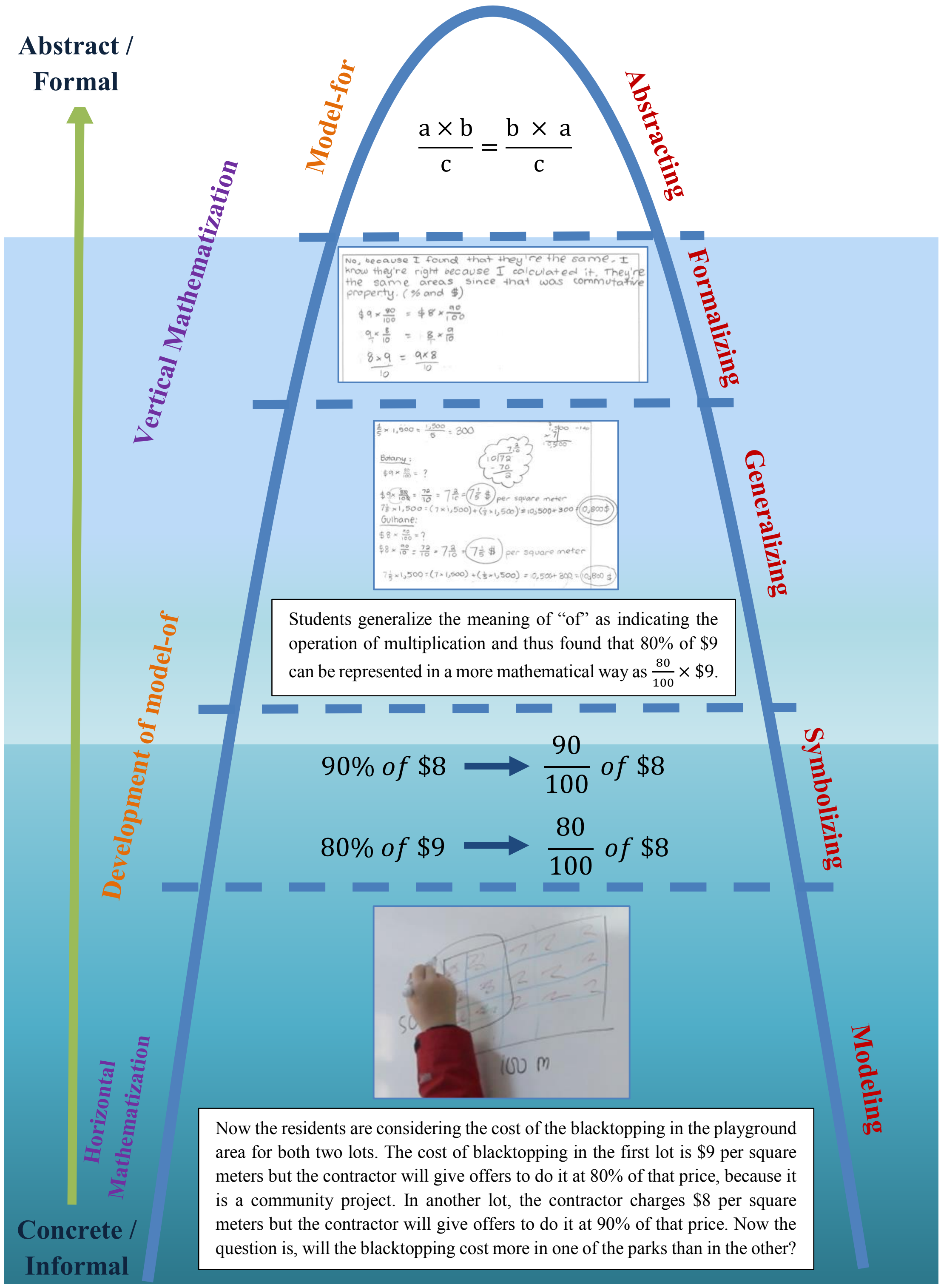


Figure 5.5. Mathematizing Process of Learning Multiplication of Fraction with Natural Number in Iceberg Metaphor – Comparing the Cost of Blacktopping Context

5.5. Implications of the Study

The study examining the mathematizing processes through instructional activities and HLT on fraction multiplication developed by using RME approach has important implications for mathematics education. By utilizing an RME approach, which emphasizes connections between mathematical concepts and real-life situations, students could develop a deeper understanding of mathematical concepts (Van den Heuvel-Panhuizen & Robitzsch, 2017). The study's findings suggested the importance of providing students with a variety of instructional activities and tasks that allowed them to construct their own understanding of mathematical concepts and processes. The use of HLT could serve as a guide for teachers to identify key learning goals and the necessary steps to achieve them. These findings could be beneficial in improving mathematics teaching and learning by providing insight into effective instructional strategies that promote the development of mathematical understanding and proficiency among students.

Furthermore, the use of HLT in this study allowed for a more structured and organized approach to teaching and learning fractions multiplication. The trajectory served as a guide to help teachers plan and sequence tasks and activities in a way that lays a foundation for future learning. This approach is consistent with the notion of scaffolding by Vygotsky (1978), which emphasized the importance of support and guidance in the learning process.

The study also highlighted the importance of establishing connections between different mathematical concepts by building on prior knowledge. This approach is consistent with constructivist learning theories, which suggest that learning is more effective when it connects to personal experiences, existing knowledge, and social contexts (Von Glasersfeld, 1989). The use of real-life situations, as advocated by the RME approach, provided a means of connecting mathematical concepts to meaningful contexts, thereby facilitating engagement and motivation.

Besides, the study suggested that the development of mathematical proficiency through the RME approach is dependent on the quality of instructional design and its alignment with the goals and objectives of the curriculum (Hiebert & Lefevre, 1986). The use of HLT enabled the integration of both curricular goals and mathematical concepts into a cohesive learning experience that promotes an understanding of how mathematical concepts relate to real-world problems. This approach could be beneficial for both teachers and students in promoting a deeper understanding of mathematical concepts.

In addition to emphasizing the importance of instruction design and alignment with curriculum goals, the study also found that the use of models of fractions can support the conceptual growth of students (Bentz & Baur, 2019). Specifically, the number line, ratio table, and area model/array were found to be effective in promoting a transition from a model of fractions to a model for fractions perspective. This approach is consistent with the RME theory, which emphasizes the importance of connecting mathematical concepts to real-life situations.

The number line model is particularly helpful in understanding fraction multiplication (Hiebert & Lefevre, 1986; Lee & Lee, 2023), as it provides a visual representation of multiplying fractions in terms of repeated steps. The ratio table model allows students to examine the fractional relationship between numbers and analyze how changes in one quantity affect another (Cramer & Wyberg, 2009). In the area model/array, students partition a shape into equal parts to demonstrate the connection between multiplication, division, and fractions (Cramer & Henry, 2002). Moreover, the use of these models encourages classroom group discussion as students must articulate their understanding of these models to their peers. The use of these fraction models presented great potential for supporting the conceptual growth of students throughout the learning of fraction multiplication. As other mathematical concepts depend on understanding fractions, this perspective has the potential for being equally effective teaching other mathematical concepts.

Despite the limitations of this study, such as the small sample size and the fact that the findings were not necessarily generalizable, its implications for educational practice could be considered as significant. The study focused on the use of the number line, ratio table, and array model as representations of fractions, which proved to be effective in supporting conceptual growth in fraction multiplication. This suggested that these pedagogical choices could benefit other mathematics programs that seek to promote understanding through constructivist approaches.

For instance, the results of this study highlighted the importance of establishing connections between different mathematical concepts instead of treating them as separate strands. Instead, the focus should be on progressing from one meaningful and useful situation to the next, forming a network of related situations that could serve as the foundation for building mathematical understanding. This approach could facilitate what is referred to as vertical mathematization, where students build on their existing knowledge base to understand more complex mathematical concepts (Greeno, 1991; Forman & Fyfe, 1998).

Learning activities, which ideally should not be limited to a particular learning strand, should offer the opportunity for discussion of familiar situations to all of the students participating in the activities. These settings were chosen to make certain that the context-bound mathematical knowledge generated through horizontal mathematization is constructive and interconnected with one another. In addition to this, society stands to benefit from this knowledge, as it makes vertical mathematization operations simple to carry out. The vast majority of students, but not all of them, should have the opportunity to go through vertical mathematization, which will allow them to discover how exciting and challenging formal mathematics can be.

As was mentioned earlier, the learning programs that are a part of this research offerred a wide range of possibilities for the development of mathematical education. The number line, ratio table, and area/array models can all be placed in realistic ways, which enabled students to apply their understanding of fractions in real-world situations. In addition to this, the models offerred a doable strategy for formalization as well as the development of a sense of numbers.

The understanding of percentages, ratios, and natural numbers is intertwined when learning about fractions through number lines, ratio tables, and area/array models. The researcher believes the learning programs have many characteristics that should be considered when renewing fraction programs. Some of these characteristics included the use of measuring as a key context, the development of number line, ratio table, and array as models, and the characterization of equivalent fractions as fractions that share their position. While the researcher assumed that other fraction programs have similar connections to other learning strands and can be used to help students develop number sense, the researcher believes that the learning programs have these characteristics.

Each and every instructional program that is generated is then put through new developmental research reflections, which served as the foundation for future developments. During the course of the development process, new ideas come to light that can serve as tools for further development work (Gravemeijer, 1994). As a result, this study has the potential to act as a potential source of insights for the development of mathematics instruction in the future.

In conclusion, the study highlighted the importance of connecting mathematical concepts to real-life situations and the use of instructional design and its related HLT. Additionally, the findings suggest that the use of models of fractions can be effective in supporting the development of students' conceptual understanding, especially in the learning of fraction multiplication. By emphasizing these perspectives, mathematics education can be more effective in providing students with a deeper understanding of mathematical concepts and their connection to real-world situations. The study's approach can be generalized to other mathematical concepts and used as a guide for future research.

The findings and conclusions of this study held great importance for a wide range of people in the field of mathematics education. For mathematics teachers, the results of this study provided insights into how to design and deliver instruction that will help students better understand and learn fraction multiplication. Moreover, for primary school teachers and preservice teachers, this study presents an opportunity to reflect on their own instructional practices and evaluate how they support students in their learning.

Additionally, teacher educators and curriculum developers can use the findings and conclusions of this study to improve their training programs and curriculum materials. This will enable them to better prepare future teachers with the necessary knowledge and skills to help their students understand and learn fraction multiplication. Furthermore, educational stakeholders, such as policymakers and school administrators, can use the results of this study to develop policies and programs that promote effective teaching practices of fraction multiplication.

Lastly, mathematics education researchers can use the findings of this study as a basis for further research. For instance, this study provides an avenue for future research on the impact of different instructional practices on students' understanding of other mathematical concepts. Moreover, it can be used as a starting point for researchers who are interested in exploring how students develop their fraction knowledge and the most effective ways to support that development.

5.6. Suggestions for Further Research

The present study provided evidence that instructional activities designed based on Realistic Mathematics Education (RME) can facilitate fifth-grade students' learning of fraction multiplication and empower them to mathematize their learning. Further research can explore the effectiveness of RME-based instructional approaches in promoting conceptual understanding of fraction multiplication among diverse student populations.

One avenue of research is to investigate the impact of RME-based instruction on students' long-term retention and transfer of learning. While this study has demonstrated the effectiveness of RME-based instruction in the short-term, it is important to investigate whether this approach can lead to lasting changes in students' understanding of fraction multiplication. Moreover, research can explore the extent to which students can apply their conceptual understanding of fraction multiplication to solve real-world problems, a central goal of the RME approach.

Additionally, follow-up studies can examine the impact of individual differences, such as students' mathematics anxiety or motivation, on their engagement and learning in RME-based instruction. While RME-based instruction has been shown to promote conceptual understanding and motivation among students, it is crucial to explore the extent to which individual characteristics may impact the effectiveness of this approach.

Another avenue of research is to investigate the role of prior knowledge and prerequisite skills in fraction multiplication. Students may struggle with fraction multiplication because they lack a solid understanding of basic fraction concepts and operations, such as fraction equivalence and addition. Research can investigate the extent to which students' prior knowledge and skills predict their success in learning fraction multiplication, and explore instructional strategies that can address any gaps in their understanding.

Another possible area of research is to investigate the effectiveness of different instructional strategies in addressing the challenges that students face when working with unfriendly fractions. For instance, this study used the number line, ratio table, and area/array model to represent fractions, which proved to be effective in promoting conceptual understanding. Further investigation can explore the effectiveness of other models or tools that could be used to support student learning in this area.

Furthermore, follow-up studies can examine the transferability of conceptual understanding of fraction multiplication to other mathematical contexts. It would be valuable to investigate whether conceptual understanding of fraction multiplication acquired through using the number line, ratio table, and array model extends to other fraction operations such as division. Additionally, the transferability of conceptual understanding of fraction multiplication to other mathematical concepts unrelated to fractions can also be explored.

Another possible direction for future research is to examine the professional development needs of teachers in addressing students' challenges in fraction multiplication. Since teachers play a crucial role in supporting students' mathematical learning, it is crucial to explore their needs in this area. Research can investigate the barriers that teachers face in teaching fraction multiplication, their beliefs and attitudes towards this concept, and the most effective approaches to providing professional development support to teachers.

In conclusion, the current study suggested that instructional activities designed based on RME can facilitate fifth-grade students' learning of fraction multiplication and empower them to mathematize their learning. Further research can deepen our understanding of the effectiveness of RME-based instruction, explore the impact of individual differences, and investigate the challenges that students face when working with unfriendly fractions. These findings have implications for improving mathematics education and promoting students' mathematical learning.

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APPENDIX

A. Lesson Plans and Student's Worksheets

LESSON PLAN

Topic	**: Multiplication of Fraction with Natural Number**
Grade	**: V**
Activity	**: Running for Fun**
Time Allocation	**: 2 × 45 minutes**
Meeting	**: 1**

A. Learning Goal

Students will develop several big ideas related to multiplication with fractions, such as fractions represent a relation, the whole matters and to maintain equivalence, the ratio of the related numbers must be kept constant.

B. Materials

Marathon route sheet (Worksheet 1), pencil or pen.

C. Prerequisites

Students should be able to:

- add, multiply and divide numbers,
- find equivalent fractions,
- convert a mixed number to a fraction,
- convert an improper fraction to a mixed number, and
- simplify fractions.

D. Description of the Activity

The context is about two cousins who trained together for this year's marathon, hoping to improve their distance from last year's marathon. The problem context is as follows.

Andrew and Bella are cousins, and they trained together for this years' a 26-km marathon, hoping to improve their distance. Last year they both ran $\frac{1}{2}$ of the route. This year Bella ran $\frac{7}{12}$ of the route and Andrew ran $\frac{5}{8}$. They know this because there are markers placed every twelfth of the route's total length, to show runners where they are and how much farther they have to go to finish. There are also eight water stations equally-spaced along the way. The last one is at the finish line. Andrew and Bella want to know how many kilometers they ran this year and how much better they did than last year.

The marathon route will be shown to the students as shown in Figure 1. The marathon route is from Middle East Technical University (Orta Doğu Teknik Üniversitesi) to an International School. The markers and water stations are given in the picture.

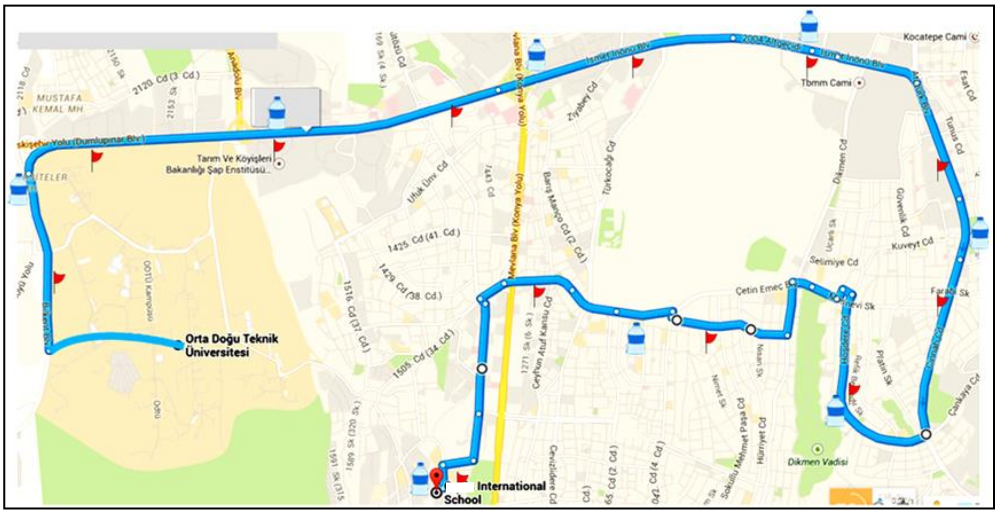


Figure 1. Marathon Route

Before students begin to solve the problem, teacher will facilitate initial discussion. Teacher will make sure that the students realize that the markers are at every twelfth of the total length, and the water stations are at every eight. Students will be given opportunities to share their preliminary thoughts then let them to solve the problem in pairs.

In order to support students' learning, teacher should encourage the students to redraw the marathon route on their posters. The discussion will focus on students' strategies to make eights and twelfths. Moreover, it will be investigated whether the students realize that every third marker will be across from a water bottle station (because 4 is a common factor of 12 and 8). Besides, will students anticipate that it will take 3 markers for every 2 water bottle stations? Teacher will let the students to explore and share their ideas in pairs. The ideas that might emerge for instance the idea of multiplication of fractions as repeated addition will be discussed in Activity 2

E. Conjectures of Students' Thinking

As the students try to solve the problems using their prior knowledge, there are some possible strategies which might emerge.

- To find $\frac{5}{8}$ of 26 or $\frac{5}{8} \times 26$, the students will try to find $\frac{1}{8}$ of 26 $(= 3\frac{1}{4})$ first, and then multiply $3\frac{1}{4}$ by 5 or using the idea of repeated addition in which $3\frac{1}{4}$ added five times. Students will use similar idea to find $\frac{7}{12}$ of 26 or $\frac{7}{12} \times 26$.
- Students may use the double number line to show repeated addition as shown in Figure 2.

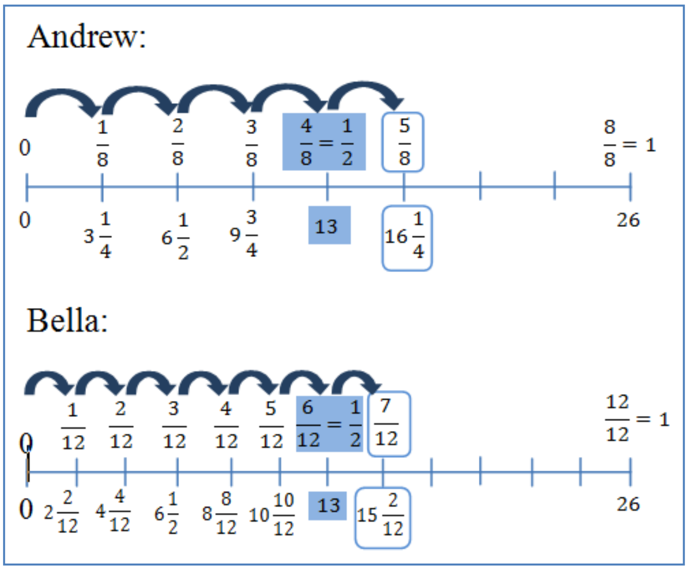


Figure 2. Double Number Line Idea

- By using distributive strategy, students may decompose $\frac{5}{8}$ into $\frac{4}{8}+\frac{1}{8}=\frac{1}{2}+\frac{1}{8}$ and then multiplying it with 26, as explained below.

  $\frac{5}{8}\times 26=\left(\frac{1}{2}+\frac{1}{8}\right)\times 26=\left(\frac{1}{2}\times 26\right)+\left(\frac{1}{8}\times 26\right)=13+3\frac{2}{8}=13+3\frac{1}{4}=16\frac{1}{4}$

  The distributive strategy as above will also be used to calculate $\frac{7}{12}\times 26$.
- By using proportional reasoning, the students will try to find $\frac{1}{8}$ of 26 by firstly find $\frac{1}{2}$ of 26 (=13), then $\frac{1}{4}$ is $6\frac{1}{2}$, and $\frac{1}{8}$ is $3\frac{1}{4}$ (Table. 3.1).

Table 1. Proportional Reasoning Idea

Andrew

Fraction Distance	Kilometers
$\frac{1}{2}$ of 26	13
$\frac{1}{4}$ of 26	$6\frac{1}{2}$
$\frac{1}{8}$ of 26	$3\frac{1}{4}$
$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$	$13+3\frac{1}{4}=16\frac{1}{4}$

Bella

Fraction Distance	Kilometers
$\frac{1}{2}$ of 26	13
$\frac{1}{4}$ of 26 or same as $\frac{3}{12}$ of 26	$6\frac{1}{2}$
$\frac{1}{12}$ of 26	$2\frac{1}{6}$
$\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$	$13+2\frac{1}{6}=15\frac{1}{6}$

Date :

Name :

WORKSHEET 1

RUNNING FOR FUN

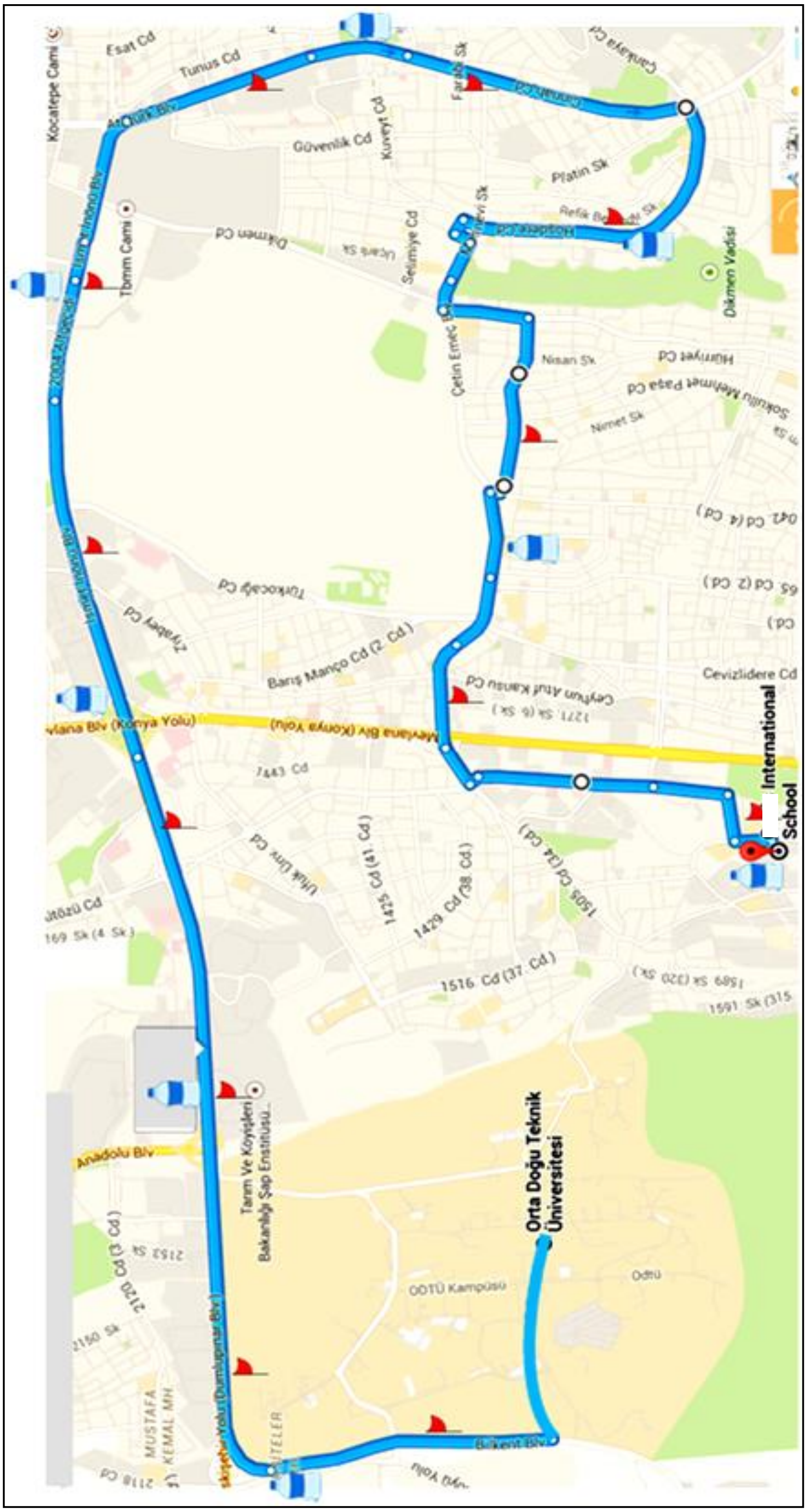
Andrew and Bella are cousins, and they trained together for this years' a 26-km marathon, hoping to improve their distance. Last year they both ran $\frac{1}{2}$ of the route. In this year training, Bella knows that she ran $\frac{7}{12}$ of the route because she counted the markers as she ran and stopped when she got to the seventh marker. Andrew really wanted to do better than he did last year. He used the water station to keep track of how he was doing. They stand at every eighth of the route. He knew when he got to the fourth water station that he had run halfway. He ran to the next one – the fifth – and then he stopped.

- How did Andrew know that the fourth water station was the halfway point?

- Did both Andrew and Bella run farther this year than they ran last year?

- How many kilometers did each of them ran this year?

- How can you assure that your conclusions are correct?


International School
Orta Doğu Teknik Üniversitesi
ODTÜ Kampüsü
Odtü
Tarım Ve Köyişleri Bakanlığı Şap Enstitüsü
Anadolu Blv
Bilkent Blv
Tbmm Camii
Kocatepe Camii
Esat Cd
Tunus Cd
Güvenlik Cd
Kuveyt Cd
Dikmen Cd
Selimiye Cd
Dikmen Vadisi
Platin Sk
Çankaya Cd
Nisan Sk
Nimet Sk
Hürriyet Cd
Sokullu Mehmet Paşa Cd
Türkocağı Cd
Ziyabey Cd
Barış Manço Cd (2. Cd.)
Ceyhun Atuf Kansu Cd
Cevizlidere Cd
Mevlana Blv (Konya Yolu)
1443 Cd
Ufuk Ünv. Cd
1425 Cd (41. Cd.)
1429. Cd (38. Cd.)
1505 Cd (34. Cd.)
1516. Cd (37. Cd.)
2120 Cd (3. Cd.)
2153 Sk
2150 Sk
2118 Cd
MUSTAFA KEMAL MH

LESSON PLAN

Topic : **Multiplication of Fraction with Natural Number**
Grade : **V**
Activity : **Math Congress - Running for Fun**
Time Allocation : **2 × 45 minutes**
Meeting : **2**

A. Learning Goals

- Students will share their ideas in solving problem especially in decomposing numbers and using partial products.
- Students will revisit their strategies discussed in the math congress and use the idea of decomposing numbers and partial products to solve the string of Mini lesson.

B. Materials

Students work from Activity 1, Worksheet 2 (Mini lesson).

C. Description of the Activity

In this activity, the students will share their ideas and strategies in solving problem in activity 1. As the students discuss their strategies, the next discussion will be structured with some considerations.

- What big ideas are likely to come up for discussion?
- What areas of confusion might arise?
- What pieces of work might provoke the students in which it will allow them to come to a generalization?

The Math Congress can be started with a piece of student's work which uses repeated addition strategy (or divided the 26 by 8 and multiplied by 5). A decomposing strategy can also be discussed, for instance decomposing $\frac{5}{8}$ into $\frac{1}{2}+\frac{1}{8}$ and then multiply the pieces using landmark fractions, and then put the partial product together. In discussing this, student's work can be used. Moreover, if any students have noticed that every third, sixth and ninth markers are across from second, fourth and sixth water stations, the discussion on why it happens will guide the students to the idea of fractions equivalence.

The Math Congress should not only be the repetition of the discussion of students' strategies. Rather, it is a moment to focus on several big ideas. Besides, to allow students to revisit and extend the big ideas discussed in the Math Congress, a set of string of related problems called Mini lesson 1 (Figure 3) will be given to the students. In this Mini lesson, the idea of fractions as operator will be emerged. Double number line can be used by students to solve the problem. The numbers in the problems have been carefully chosen to encourage the students to decompose the fractions and to use the distributive property as they may think the idea of decomposing $\frac{5}{8}$ into $\frac{1}{2}+\frac{1}{8}$.

String of related problems:

$\frac{1}{2} \times 36$

$\frac{1}{4} \times 36$

$\frac{1}{8} \times 36$

$\frac{5}{8} \times 36$

$\frac{7}{8} \times 36$

$\frac{5}{8} \times 48$

Figure 3. Mini lesson 1: Fractions as Operator

D. Conjectures of Students' Thinking

It is expected that through students' own contribution in solving problem in Activity 1, several big ideas will emerge:

- Fractions which are difficult to work with can be decomposed into unit fractions.
- Equivalent of fractions, for instance: $\frac{1}{2} = \frac{4}{8} = \frac{6}{12}$.
- Multiplication strategies for fractions can be built through equivalent of fractions, for instance $\frac{5}{8} = 5 \times \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$.
- The whole matters. When students compare whether $\frac{7}{12}$ is more than $\frac{5}{8}$, they cannot always just compare the numerators or the denominators.

- The distributive property holds for multiplication over addition for fractions. The partial product of $\frac{1}{2} \times 26$ and $\frac{1}{8} \times 26$ can be used to find the product of the whole, $\frac{5}{8} \times 26$.
- Through proportional reasoning, the students may determine that if $\frac{1}{8} \times 26 = 3\frac{1}{4}$, then $\frac{5}{8}$ of 26 = $5 \times 3\frac{1}{4} = 16\frac{1}{4}$.

Date :

Name :

WORKSHEET 2

MINI LESSON "FRACTIONS AS OPERATOR"

1. $\frac{1}{2} \times 36 =$

2. $\frac{1}{4} \times 36 =$

3. $\frac{1}{8} \times 36 =$

4. $\frac{5}{8} \times 36 =$

5. $\frac{7}{8} \times 36 =$

6. $\frac{5}{8} \times 48 =$

7. $\frac{3}{4} \times 15 =$

8. $\frac{3}{4} \times 15 =$

9. $\frac{3}{4} \times 15 =$

LESSON PLAN

Topic : **Multiplication of Fraction with Natural Number**
Grade : **V**
Activity : **Training for Next Year's Marathon**
Time Allocation : **2 × 45 minutes**
Meeting : **3**

A. Learning Goal

Students will use landmark fractions and partial products when multiplying a fraction by a natural number.

B. Materials

Worksheet 3, pencil or pen.

C. Prerequisites

Students should be able to:

- add, multiply and divide numbers,
- find equivalent fractions,
- convert a mixed number to a fraction,
- convert an improper fraction to a mixed number, and
- simplify fractions.

D. Description of the Activity

In this activity, to begin the Training for Next Year's Marathon problem, the following story will be delivered.

> For next year's marathon competition not only Andrew and Bella who do the training. Some members of the club are also doing the training. They train in a park that has a 3-kilometers running track with markers to indicate the part of the track that has been completed. Several of the members decide to keep the record of their result as follows.

Training Records			
	Minutes	Circuit of Track Completed	Rate (Minutes per Circuit)
Alex	120	4	
Ethan	60	3	
John	45	3	
Elizabeth		2	30
Benjamin		1	20
Olivia		$\frac{1}{2}$	18
Emma		$\frac{1}{4}$	20
Isabella		$\frac{3}{4}$	20
James		$1\frac{1}{2}$	20
Rafa		$2\frac{3}{4}$	30

Table 2. Training Record

Teacher will do the first problem (Alex's rate) then ask the students to complete the chart individually. To find the rate, students will be encouraged to draw pictures and will be guided to realize that the size of the whole matters, and in all cases it is one circuit.

As the students solve the problem, encourage them to notice the relationships that appear in the data in which if the ratio kept constant, equivalence is maintained. Doubling and halving may emerge for a discussion. Students may also find the relation that 'a runner may have a faster rate because s/he ran for fewer minutes'.

E. Conjectures of Students' Thinking

In solving the problems, several big ideas and strategies are likely to emerge.

- Students may find that in order to divide (by fractions), they are multiplying (by the multiplicative inverse of fraction) to find the rate.

- Some students may notice the numbers relationships and use proportional reasoning to complete the chart. For instance, the students might see since Alex ran twice as many minutes as Elizabeth and completed twice as much distance as Elizabeth, then his rate is the same as Elizabeth's rate: 30 minutes for one circuit.
- Partial quotients may also be used, for instance for the case of Isabella, she did $\frac{3}{4}$ of the circuit in 15 minutes. Then, $\frac{3}{4}$ will be divided into 3 pieces and each piece was 5 minutes. So, the whole is 20 which we add 15 and 5, because that is $\frac{3}{4} + \frac{1}{4}$.
- Some may use the relationship of multiplication and division, for instance in the case of Elizabeth, the students may find the minutes first namely 2×30 is 60. Alex ran twice as far. Then they will think about $4 \times ? = 120$, and the result will be 30, too.
- Some students may use the double number line model.
- Others may use the chart as ratio table in which the arrows will show the relations of the rates.

Date/Class :

Name :

WORKSHEET 3

TRAINING FOR NEXT YEAR'S MARATHON

Training Record			
Name	Minutes	Circuit of Track Completed	Rate (Minutes per Circuit)
Alex	120	4	…
Ethan	60	3	…
John	45	3	…
Elizabeth	…	2	30
Benjamin	…	1	20
Olivia	…	$\frac{1}{2}$	18
Emma	…	$\frac{1}{4}$	20
Isabella	…	$\frac{3}{4}$	20
James	…	$1\frac{1}{2}$	20
Rafa	…	$2\frac{3}{4}$	30

LESSON PLAN

Topic : **Multiplication of Fraction with Natural Number**
Grade : **V**
Activity : **Math Congress – the Marathon Training Results**
Time Allocation : **2 × 45 minutes**
Meeting : **4**

A. Learning Goal

Students will share their ideas in solving problem in Activity 3 especially about the patterns on the chart and ideas related to multiplication of fraction with natural number.

B. Materials

Students work from Activity 3.

C. Description of the Activity

In the beginning, teacher can ask some questions to the students for a preliminary discussion, such as:

- Now the training chart is complete, what relationships do you see and why do you think those relationships occurred?
- Which problems are solved with division and which with multiplication?

After sufficient discussion time, the big ideas that emerge from students' strategies in solving the problem in Activity 3 can be discussed with some guiding questions, such as:

- What is the whole?
- Can proportional reasoning be used to determine equivalent rates? That is, can multiplying both the dividend (minutes) and divisor (circuits) by the same number helpful in finding solutions?

Proportional reasoning is the heart of this problem, which maintains equivalence while the ratio is kept constant. The students are expected to notice that Isabella ran $\frac{3}{4}$ of the track in 15 minutes and Emma ran $\frac{1}{4}$ of the track in 5 minutes, and the dividend and divisor have both tripled. If the students struggle with Olivia's and Emma's cases as they have different rates, they are expected to draw the track and mark out the fractions on the circuit. Moreover, it is expected that the students will recognize that Emma ran $\frac{1}{4}$ of the track in 5 minutes and Isabella ran $\frac{3}{4}$ of the track in 15 minutes, and both the dividend and divisor have tripled.

Date/Class :

Name :

WORKSHEET 4

MINI LESSON "FRACTIONS AS OPERATOR"

1. $\frac{1}{2} \times 44 =$

2. $\frac{1}{4} \times 44 =$

3. $\frac{1}{8} \times 44 =$

4. $\frac{5}{8} \times 44 =$

5. $\frac{3}{8} \times 44 =$

6. $\frac{5}{12} \times 28 =$

7. $\frac{7}{12} \times 28 =$

8. $\frac{11}{12} \times 28 =$

9. $\frac{5}{8} \times 52 =$

LESSON PLAN

Topic : Multiplication of Fractions
Grade : V
Activity : Exploring Playgrounds and Blacktop Areas
Time Allocation : 2 × 45 minutes
Meeting : 5

A. Learning Goal

Students will solve the problem related to multiplication of fractions.

B. Materials

Worksheet 5, pencil or pen.

C. Prerequisites

Students should be able to:

- add, subtract, multiply and divide numbers,
- find equivalent fractions,
- convert a mixed number to a fraction,
- convert an improper fraction to a mixed number, and
- simplify fractions.

D. Description of the Activity

Students will try to solve problem given in the worksheet 5. They will solve the problem individually and their works will be discussed in the Math Congress in Activity 7. The problem is as follows.

There are two lots that will be changed into small parks in Cankaya area. Two of them are measures 50 meters by 100 meters. For Botany Garden, the residents agreed that $\frac{3}{4}$ of the lot will be devoted to a playground for children and then $\frac{2}{5}$ of that playground will be covered by blacktop, so children can play basketball and kickball and games like that. Another lot, the Gulhane Garden, the residents decided to use $\frac{2}{5}$ of the lot for a playground then $\frac{3}{4}$ of that will be blacktop.

The problem above has been carefully designed in which the array model is expected to emerge to guide the student in solving multiplication of fractions problem. The numbers in the problem are also carefully chosen to develop the commutative property: $\frac{2}{5} \times \frac{3}{4} = \frac{3}{4} \times \frac{2}{5}$.

Through this activity, it is expected that the students will develop the array model to represent the relationship of playground to lot, blacktop to playground, and blacktopped-playground to lot.

In solving the problem, teacher will encourage students to present a clear idea about their strategies. It is expected that they can generalize the idea and then proof their generalization. A discussion with the students will focus on the following ideas:

- The meaning of the word "of" as the sign of multiplication of fractions.
- The commutative property of multiplication as it also applies to fractions.
- The relationship of blacktopped area to the total area of the lot represented by the array model.

E. Conjectures of Students' Thinking

It is expected that students will discover that the blacktop areas are equivalent. Besides, it is also expected that the idea of equivalence versus congruence will come up. Although the areas are equivalent, depending on how they are drawn by students, they may not be congruent. Below are strategies and challenges that might appear in students' solution.

- Students will cut both lots fourths vertically and fifths horizontally and then shade or color the parts in which indicate the blacktopped area, $\frac{3}{4}$ of $\frac{2}{5}$ or $\frac{2}{5}$ of $\frac{3}{4}$. With this strategy, the ratio of the array of the blacktopped area (3×2) to the array of the lot

(4 × 5). So, the blacktopped area of two lots will congruent as shown in Figure 4 below.

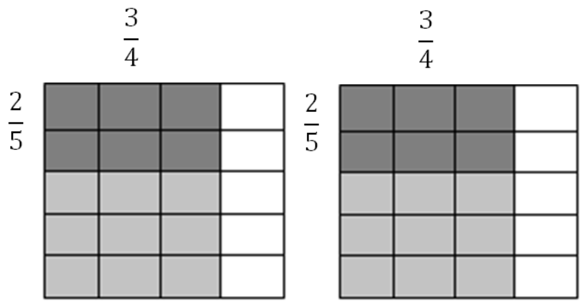


Figure 4. Blacktop Areas of Two Lots (Strategy 1)

- Students will cut one lot into fourth horizontally then shade $\frac{3}{4}$ indicating the playground, then marking fifths of that area vertically and shade $\frac{2}{5}$ of it the show the blacktop area. Similar way for the other lot, students will first cut the lot into fifths horizontally then shade $\frac{2}{5}$ indicating the playground, then marking fourths of that area vertically and shade $\frac{3}{4}$ of it the show the blacktop area (Figure 5). The thing that might challenge students through this strategy is that the areas are equivalent but non-congruent and students need to find a way to compare them.

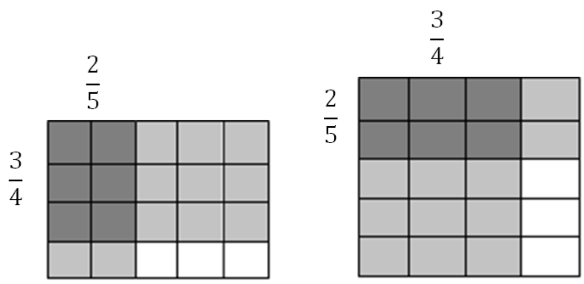


Figure 5. Blacktop Areas of Two Lots (Strategy 2)

- Students perhaps cut the fifths or fourths vertically (or horizontally). Students will find overlapping part but they might struggle to determine the fractional part.

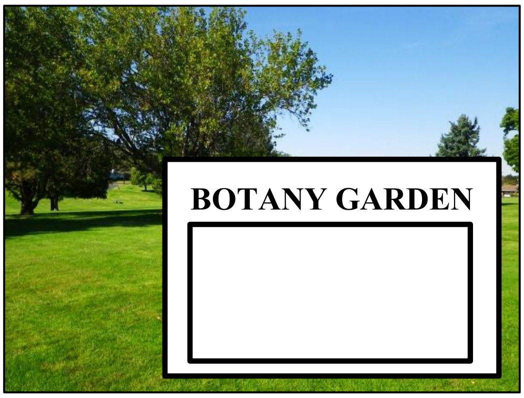
- Students may use the dimensions of the lots which is 50 meters × 100 meters. This strategy may give result 37.5 meters × 40 meters, and 75 meters × 20 meters. Using this strategy, the students will find that the areas are equivalent but not congruent.

Date/Class :

Name :

WORKSHEET 5

EXPLORING PLAYGROUNDS AND BLACKTOP AREAS



There are two lots, Botany and Gulhane gardens, that will be changed into small parks in Cankaya area. Two of them are measures 50 meters by 100 meters. For Botany Garden, the residents agreed that $\frac{3}{4}$ of the lot will be devoted to a playground for children and then $\frac{2}{5}$ of that playground will be covered by blacktop, so children can play basketball and kickball and games like that. Another lot, the Gulhane Garden, the residents decided to use $\frac{2}{5}$ of the lot for a playground then $\frac{3}{4}$ of that will be blacktop.

- Is there more blacktop space in one lot than in other lot?

- How would you convince the people in the two neighborhoods that your conclusion is correct?

LESSON PLAN

Topic : **Multiplication of Fractions**
Grade : **V**
Activity : **Math Congress - Comparing Blacktop Areas**
Time Allocation : **2 × 45 minutes**
Meeting : **6**

A. Learning Goals

- Students will share their ideas in solving problem in Activity 6 especially several big ideas related to multiplication of fractions.
- Students will use array model to solve the multiplication of fractions problem.

B. Materials

Students work from Activity 6, worksheet 6, pencil or pen.

C. Description of the Activity

In this activity, the students will share their strategies in the discussion of Math Congress. The questions that can be asked to the students, in the discussion, to develop their understanding about multiplication with fractions, namely:

- How is one strategy connected to another?
- What are the big ideas behind the problem?
- How the problem could generate a generalization about multiplication with fractions?

  Moreover, the big ideas which will be discussed in the Math Congress, namely:

- The meaning of "of" as signifying multiplication with fractions
- The commutative property of multiplication which also applies to fractions
- The relationship of blacktopped area to the total area of the lot represented by the array model
- The changing whole
- Equivalence of fractions, if some students simplify $\frac{6}{20}$ into $\frac{3}{10}$.

- The standard algorithm for multiplication of fractions is that the product is the relationship of the array created by the numerators over the array created by the denominator.

The activity will be continued by giving the students the Mini lesson 3 (Figure 6) which consists of a string of related problems of multiplication of fractions. This Mini lesson 3 will allow the students to use the array model to solve the problem.

String of related problems:

$$\frac{1}{3} \times \frac{1}{5}$$

$$\frac{1}{3} \times \frac{3}{5}$$

$$\frac{2}{3} \times \frac{3}{5}$$

$$\frac{1}{6} \times \frac{3}{5}$$

$$\frac{5}{6} \times \frac{3}{5}$$

$$\frac{5}{6} \times \frac{4}{5}$$

$$\frac{7}{8} \times \frac{3}{4}$$

Figure 6. Mini lesson 3: Multiplication of Fractions

Date/Class :

Name :

WORKSHEET 6

MINILESSON "MULTIPLICATION OF FRACTIONS"

1. $\frac{1}{3} \times \frac{1}{5} =$

2. $\frac{1}{3} \times \frac{3}{5} =$

3. $\frac{2}{3} \times \frac{3}{5} =$

4. $\frac{1}{6} \times \frac{3}{5} =$

5. $\frac{5}{6} \times \frac{3}{5} =$

6. $\frac{5}{6} \times \frac{4}{5} =$

7. $\frac{5}{7} \times \frac{14}{15} =$

8. $\frac{3}{7} \times \frac{15}{17} =$

9. $\frac{7}{8} \times \frac{3}{4} =$

LESSON PLAN

Topic	**: Multiplication of Fractions, Commutative property of Multiplication of Fractions**
Grade	**: V**
Activity	**: Comparing the Cost of Blacktopping**
Time Allocation	**: 2 × 45 minutes**
Meeting	**: 7**

A. Learning Goal

Students will extend their investigation with the commutative property of multiplication of fractions to percentages and decimals.

B. Materials

Worksheet 7, pencil or pen.

C. Description of the Activity

The activity 8 will be started with the following problem in which the students are asked to find the cost of blacktopping areas.

> Now the residents are considering the cost of the blacktopping in the playground area for both two lots. The cost of blacktopping in the first lot is $9 per square meters but the contractor will give offers to do it at 80% of that price, because it is a community project. In another lot, the contractor charges $8 per square meters but the contractor will give offers to do it at 90% of that price. Now the question is, will the blacktopping cost more in one of the parks than in the other?

Through this problem, students will examine ways of multiplying with equivalent form of fractions – percentages and decimals – and extend their work with rational numbers. Students will work individually and write their solutions to be discussed in the Math Congress in Activity 9.

D. Conjectures of Students' Thinking

As students go on to the problem of comparing the cost of blacktopping, several strategies are likely to emerge:

- Before comparing the cost, the students will calculate the area of blacktopping for each lot. The calculation depends on their drawing in Activity 6, whether they will calculate $\frac{3}{4}$ of 50 and $\frac{2}{5}$ of 100; or $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100. In calculating $\frac{2}{5}$ of 50 and $\frac{3}{4}$ of 100, students will probably not have difficulty than calculating $\frac{3}{4}$ of 50. In calculating $\frac{3}{4}$ of 50, students might start with what they know, for instance by first find $\frac{1}{2}$ of 50 (=25) and $\frac{1}{2}$ of 25 (=12.5) which is same as $\frac{1}{4}$ of 50, and then they can calculate $\frac{3}{4}$ of 50 as 37.5 meters. Students might find the commutative properties that underlies the relationship of both blacktopping in two lots:
  37.5 meters × 40 yards = 75 meters × 1,500 yards = 1,500 square meters
- After students find the blacktopping areas of two lots, students may multiply the area with the price per meters and then subtract out the discount. For instance, for the first blacktopping area with \$9 per square meters and offers 80% of the price, students might multiply \$9 by 1,500 to determine the full price of blacktopping which is \$13,500; then to find 20% of \$13,500, students will use landmark of fraction in which 20% equal to $\frac{1}{5}$ and then calculate $\frac{1}{5}$ of \$13,500 (the discount, by dividing by 5, \$2,700); and subtract that discount to get the final price of \$10,800. Similarly, to find the price of blacktopping in other lot uses \$9 per square meters and $\frac{1}{10}$ to calculate the discount.
- Some students will directly include the discount in the calculation and use decimal or fraction forms to show the percentages (i.e., 80% = 0.8 = $\frac{8}{10}$; 90% = 0.9 = $\frac{9}{10}$). Then, calculate $0.8 \times \$9\ \times\ 1{,}500\, m^2$ for the cost of first blacktopping area and calculate $0.9 \times \$8\ \times\ 1{,}500\ square\ meters$ for the cost of second blacktopping

area. To find the calculation, the students might decompose the percentages or fractions through associative property: $\left(8 \times \frac{1}{10}\right) \times 9 \times 1{,}500 = 8 \times \left(\frac{1}{10} \times 9\right) \times 1{,}500$

- Other students may thing that there is no need to include the area as they found it is equivalent. They may just use $0.8 \times \$9$ and $0.9 \times \$8$. In this case, they might be challenged about why 80% of 9 = 90% of 8. It is expected that the students will find the associative property underlies the equivalence:

$$8 \times \left(10 \times \frac{1}{100}\right) \times 9 = 9 \times \left(10 \times \frac{1}{100}\right) \times 8$$

Date/Class :

Name :

WORKSHEET 7

COMPARING THE COST OF BLACKTOPPING

Continuing previous problem about 'exploring playgrounds and blacktop areas', now the residents are considering the cost of the blacktopping in the playground area for both two lots. The cost of blacktopping in the first lot is \$9 per square meters but the contractor will give offers to do it at 80% of that price, because it is a community project. In another lot, the contractor charges \$8 per square meters but the contractor will give offers to do it at 90% of that price. Now the question is, will the blacktopping cost more in one of the parks than in the other?

LESSON PLAN

Topic	**: Multiplication of Fractions, Commutative Property of Multiplication of Fractions, Percentages and Decimals**
Grade	**: V**
Activity	**: Math Congress – Comparing the Cost of Blacktopping**
Time Allocation	**: 2 × 45 minutes**
Meeting	**: 8**

A. Learning Goals

- Students will use interchanging numerators (or denominators) to derive the product of two fractions.
- Students will share their ideas in solving problem in Activity 8 especially related to the commutative property and associative property of multiplication with fractions, percentages and decimals.

B. Materials

Students work from Activity 8, worksheet 8, pencil or pen.

C. Description of the Activity

The activity will be started by giving the students the Mini lesson 4 (Figure 8) which consists of a string of related problems of interchanging numerators to multiply with fractions. This Mini lesson 4 will allow students to make the problem 'friendlier', for instance $\frac{2}{3} \times \frac{3}{5} = \frac{3}{3} \times \frac{2}{5}$. This is string is designed to develop that strategy.

String of related problems:

$\frac{1}{3} \times \frac{1}{5}$

$\frac{2}{3} \times \frac{1}{5}$

$\frac{3}{5} \times \frac{2}{3}$

$\frac{2}{5} \times \frac{3}{3}$

$\frac{1}{5} \times \frac{1}{7}$

$\frac{2}{5} \times \frac{5}{7}$

$\frac{5}{5} \times \frac{2}{7}$

$\frac{3}{5} \times \frac{2}{3}$

Figure 8. Minilesson 4: Interchanging Numerators

In the Math Congress, students will discuss several strategies that emerge in solving problem in Activity 8. The variety of strategies that might bring to a discussion, namely:

- A 20% discount means that the price is now 80% of the original price.
- Using landmark or 'friendly' fractions can be useful strategy, for instance $10\% = \frac{1}{10}$, $20\% = \frac{1}{5}$, $25\% = \frac{1}{4}$ and $50\% = \frac{1}{2}$
- In determining equivalent fractions, percentages and decimals, rations must be kept constant, such as $80\% = 0.80 = 0.8 = \frac{8}{10} = \frac{4}{5}$
- Commutative and associative properties hold for multiplication of decimals and percentages, as well as for multiplication of fractions.

Regarding the cost of blacktopping areas, to encourage the students to generalize, two different representations can be helpful.

- Students may use arrays to determine the equivalence. This array model will provide opportunity for students to revisit the commutative property.

  $80\% \times 9 = 90\% \times 8$ because $\frac{72}{90} = \frac{72}{90}$ (Figure 7).

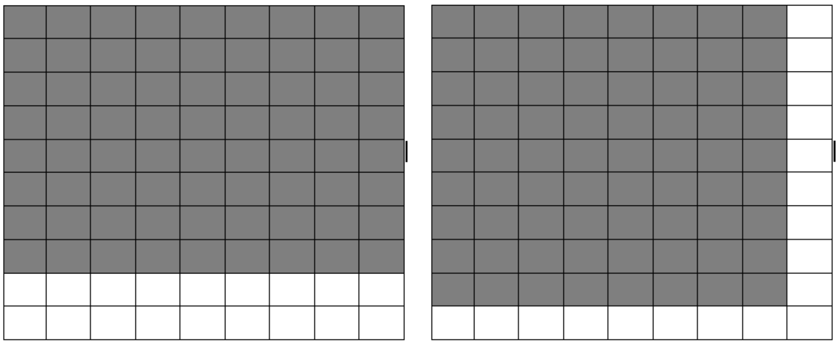


Figure 7. Array Model Representing $80\% \times 9 = 90\% \times 8$

The numeric representation such as landmark percentages and the area of the blacktop to compute and compare the costs. For instance, students may argue that $80\% \times 9 = 90\% \times 8$ because $80\% \times 9 = \frac{8}{10} \times 9 = \left(8 \times \frac{1}{10}\right) \times 9 = 8 \times \left(\frac{1}{10} \times 9\right)$. This representation will provide students the opportunity to explore the associative strategy.

Date/Class :

Name :

WORKSHEET 8

MINILESSON "INTERCHANGING NUMERATORS"

1. $\frac{1}{3} \times \frac{1}{5} =$

2. $\frac{2}{3} \times \frac{1}{5} =$

3. $\frac{3}{5} \times \frac{2}{3} =$

4. $\frac{2}{5} \times \frac{3}{3} =$

5. $\frac{1}{5} \times \frac{1}{7}$ =

6. $\frac{2}{5} \times \frac{5}{7}$ =

7. $\frac{5}{5} \times \frac{2}{7}$ =

8. $\frac{7}{9} \times \frac{3}{14}$ =

9. $\frac{3}{14} \times \frac{7}{12}$ =

CURRICULUM VITAE

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EDUCATION			
	Degree of Graduation	**Major**	**Institution**
Ph.D	3.88 / 4.00	Mathematics Education	Middle East Technical University, Türkiye
M.Sc	3.76 / 4.00	Mathematics Education	Sriwijaya University, Indonesia and Utrecht University, the Netherlands
B.Sc	3.80 / 4.00	Mathematics Education	State University of Jakarta, Indonesia

WORK EXPERIENCES	
Ankara, Türkiye July 2014 – present	Senior Researcher Statistical, Economic and Social Research and Training Centre for Islamic Countries (SESRIC) Subsidiary Organ of the Organisation of Islamic Cooperation (OIC)
Tangerang, Indonesia March 2012 – September 2012	Mathematics Lecturer and Researcher Surya College of Education
Tangerang, Indonesia March 2012 – September 2012	Teacher Trainer for GASING Learning Model Surya Institute
Jakarta, Indonesia August 2011 – February 2012	Mathematics Lecturer Sampoerna School of Education

Cikarang, Indonesia 2008 – 2009	Mathematics High School Teacher International Islamic Boarding School
Jakarta, Indonesia 2008	Mathematics High School Teacher Al-Azhar Syifa Budi
Bekasi, Indonesia 2007 - 2008	Mathematics High School Teacher SMA Mutiara 17 Agustus
Jakarta, Indonesia 2005 – 2008	Mathematics Book Editor Erlangga Publisher

THESES

- Master Thesis
  Shanty, Nenden Octavarulia. 2011. Design Research on Mathematics Education: Investigating the Progress of Indonesian Fifth Grade Students' Learning on Multiplication of Fractions with Natural Numbers. International Master Programme on Mathematics Education (IMPoME), in Collaboration between Sriwijaya University, Indonesia, and Utrecht University, the Netherlands. http://www.fisme.science.uu.nl/en/impome/theses_group_2010/thesis_Nenden.pdf

- Bachelor Thesis
  Shanty, Nenden Octavarulia. 2007. Readability Analysis of Mathematics Teaching Materials Using the Indonesian Realistic Mathematics Education (PMRI) Approach on Number Subject in Grade 2 Elementary School in Jakarta (Unpublished bachelor thesis). State University of Jakarta, Jakarta, Indonesia.

PUBLICATIONS

- Ozturk, Ahmet & Shanty, Nenden Octavarulia. (2021). Implementation of the Tobacco Questions for Surveys (TQS) in Selected OIC Member Countries: Evidence for Action. Ankara, Turkiye: SESRIC.
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- Shanty, Nenden Octavarulia. (2016). Sources of Lecturer Assistants' Teaching Anxiety in Teaching Instructional Technology Course: A Qualitative Interview

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- Shanty, Nenden Octavarulia., de Haan, Dede & Ilma, Ratu. (2011). Design Research on Mathematics Education: Investigating the Progress of Indonesian Fifth Grade Students’ Learning on Multiplication of Fractions with Natural Number. Indonesian Mathematical Society – Journal on Mathematics Education, Vol.2.No.2, ISSN: 2087-8885.
http://jims-b.org/wp-content/uploads/2013/11/Full-IndoMS-JME-22-Nenden.pdf

LANGUAGES	
Bahasa Indonesia	Native speaker
English	Fluent in reading, speaking, writing, and listening
Turkish	Basic